Basic Differentiation Rules and Rates of Change

1. Welcome to basic differentiation rules and rates of change. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include evaluating functions, radicals as rational exponents, simplifying rational expressions, and knowing basic exponential rules. The trig skills you need are knowing the graphs of the sine and cosine functions, being able to evaluate trig functions, and solving trigonometric equations.
3. Let's get started with Calculus I Differentiation: Basic Differentiation Rules and Rates of Change. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 2.2.
4. We are going to take the derivative rules a little at a time and practice the steps before we put them all together. In part 1 we will have two rules, the first is the constant rule. The derivative of any constant is zero. We can see this by looking at a graph of a constant function. Notice that it's slope is zero. The derivative is just the slope, so if you have a line with a constant slope of zero then it's derivative will also be constantly zero.
5. The power rule is one of the rules we use most frequently. The power in the power rule refers to the type of function, power functions. That is, functions with a variable base and real number exponent. The rule states that if n is a rational number, then the function f of x equal to x to the n is differentiable at the derivative is equal to n multiplied by x to one power lower. There is a special note to this one: for f to be differentiable at zero, the integer n must be a number such that x to the n minus 1 power is defined near 0.
6. Examples of the constant function are a little boring, but we will come to them later in this lecture. Let's look at a few examples of the power rule. Let f of x equal x to the 5^{th} power at the x value 2. Our goal is the find the slope of the tangent line at that value. Remember, the slope of the tangent line is a fancy way of saying "take the derivative and evaluate it." The slope of the tangent line is the derivative. Using the power rule we bring down the 5, subtract one from the exponent, and simplify. The derivative function is f prime of x equals 5 x to the fourth. Now we can substitute x = 2 and find that the slope of the tangent line at 2 is 80. That's a pretty steep line.
7. For example two our function is f of x equals x to the 2/3rds at x equals 1. First we use the power rule to find the derivative bringing down the 2/3 and subtracting one from the exponent. After subtracting one from the exponent we have a negative exponent. It is somewhat strange evaluating functions with negative exponents, so get in the habit of always rewriting with

positive exponents. In this case the derivative will be 2 divided by 3 x to the 1/3. Substituting 1 for x we find the slope of the tangent line to be 2/3.

8. Our last example of this type is a rational function f of x equals 1 over x to the third at the x value of -2. We do not have a rule that deals with rational functions, but we can rewrite this as a power function x to the negative 3. Now taking the derivative consists of bringing down the negative three and subtracting one from the exponent. The derivative is negative three x to the negative 4. Once again, we will be evaluating so we rewrite with a positive exponent a negative 3 over x to the fourth. Finally we can evaluate when x = 2 to get -3 over 16. This gives us a series of steps that you will see repeatedly: rewrite in order to take the derivative, then take the derivative, then rewrite to evaluate, and finally evaluate. The hardest part of calculus is the algebra. Take your time in rewriting to keep the calculus as easy as possible and not get confused in the signs of the exponents with the algebra.
9. Our second part of the rules contains the constant multiple rule and the sum and difference rules. The constant multiple rule states that, just as in limits, constants can be moved to the front prior to finding the derivative. We just have to make sure to multiply them by the derivative. That is, the derivative of c times f is c times the derivative of f. The sum and difference rules are similar except for one is talking about addition whereas the other is talking about subtraction. Basically, the derivative of a sum is the sum of the derivatives and the derivative of a difference is the difference of derivatives.
10. Let's take a closer look at the constant multiple rule with some examples. Number 1 has f of x equal to 3 x to the 5^{th}. We leave the 3 in front and take the derivative of x to the 5^{th} by bringing down the 5 and subtracting one from the exponent. We now multiply the 3 times the 5 and simplify the numerator making 15 x to the fourth. Number 2 has f of x equals seventeen x squared. To find the derivative we leave the 17, take the derivative of x squared by bringing down the 2 and subtracting one from the exponent. Now multiply by the 17 to get 34 x. Number 3 is f of x equals negative five x to the negative seventh. Leave the negative 5 out front, bring down the negative 7 and subtract one from the exponent. Remember, negative seven minus one is negative eight. It seems simple, but we sometimes go to fast and neglect the easy steps. Multiplying negative five times negative seven gives positive 35 and that is multiplied by x to the negative eight. In general, if the problem starts with negative exponents it is ok to end with negative exponents as well.
11. Think about what we can do now. We can take derivatives of constant terms, power functions, constant multiples and sums and differences. This means we can take derivatives of any polynomial and many root functions using the rules rather than the limit definition of derivatives.
12. Let's now use all the rules to find some derivatives. Our first function is f of x equals 9. This is a constant function and so its derivative is zero. Number two has f of x equal to the fourth root of x. We rewrite in rational exponent notation to get x to the one-fourth, then take the derivative using the power rule, and finally rewrite with a positive exponent or in radical form.
13. Number three is f of x equals the polynomial x squared minus 2x plus 3. The sum and difference rules tell us that we can look at each term individually. The derivative of x squared brings down the 2 and subtracts one from the exponent giving 2x. The derivative of minus 2x is negative 2

times 1 x to the 1 minus 1. But 1 minus 1 is zero and x to the zero is 1 so the derivative of minus 2x is negative 2. The derivative of the constant term 3 is zero. Putting this all together we have a derivative of 2x minus 2. Problem 4 is y equals 8 minus x to the third. The derivative of 8 is zero and the derivative of x to the third is three times x squared. We must keep the subtraction so our derivative is negative 3 x squared.

14. Our fifth, and last, example of this type is another polynomial. Let f of x equal 2 x cubed minus x squared plus 3x. Take the derivative of each term individually to get 6 x squared minus 2x plus 3. In the previous example we saw that the derivative of -2x was -2 and in this example the derivative of 3x is 3. In general, the derivative of any linear term mx will always be m. That is, the slope of a linear equation or linear term is the coefficient of x.
15. Knowing the graphs of the sine and cosine function from memory is helpful in a variety of ways. In calculus, knowing the graphs can help us with the derivatives. Looking at the graph of the sine function, bottom left, we can estimate the slope of various tangent lines at the key points of the graph. Plotting these slopes as the y values we can graph the derivative of sine. Notice that the graph of the slopes of sine (the derivative of the sine function) is the cosine function. You can do a similar analysis starting with the cosine graph and finding the derivative is the negative sine function. Memorize these.
16. Using our new trig rules of differentiation we find the derivative of g of t equals pi times cosine of t by leaving the constant pi in front and multiplying it by the derivative of cosine of t giving g prime of t equals negative pi times sine of t. The second function y equals 7 plus sine of x has derivative 0 plus cosine of x which simplifies to cosine of x. These seem simplistic as I say them. Let them remain that way in all future derivatives. Each individual piece is doable and “easy.” Know the rules and know them well to help when we start the more complicated rules.
17. Some functions are given to us in a way that makes sense for the problem at hand, but not necessarily for what we need to do with it. If y equals 5 over the quantity of 2x all cubed plus 2 cosine x, we will need to simplify or rewrite before we can take the derivative as instructed. To rewrite, first cube the denominator and then write the variable with a negative exponent. In the future we will be able to deal with variables in the denominator, but we don’t have that rule yet so we must rewrite. Now that we can use the power rule we take the derivative. We use the constant multiple rule, the power rule, the sum rule and the derivative of cosine rule all for this two term function. Since the function started with positive exponents, we want to take the extra step of rewriting the derivative so that it has positive exponents as well. In this case y prime is equal to negative 15 over the quantity 8 x to the fourth minus 2 sine of x.
18. Our previous examples asked us to find the derivative. Now we are looking for the slope of the graph (also known as the slope of the tangent to the graph) at a given point. This is saying find the derivative and then evaluate at this given point. For f of x equal 8 over x squared at 2 comma 2 we will rewrite to get 8 x to the negative 2. Next we find the derivative using the power rule to be negative 16 x to the negative 3. Before we can evaluate we must rewrite with positive exponents. Get used to this, we will do it frequently. The derivative is negative sixteen over x cubed and now we can evaluate it when x equals 2 to get a slope of the graph of negative 2.
19. For the function f of t equals 3 minus 3 over the quantity 5t at the point 3/5 comma 2 we first rewrite to get the variable out of the denominator. Then take the derivative remembering that

the derivative of a constant, 3, is zero while a constant multiple, 3/5, stays put and gets multiplied by the derivative of t to the negative 1. Once we have the derivative we rewrite with positive exponents so that we can evaluate. Many students do not like fractions so don't freak out when you see that you have to evaluate at 3/5; just take it one step at a time. Evaluating the derivative we have 3 over all of 5 times 3/5 squared. Order of operations tells us to square and get 9/25 and then multiply by the 5 to get 9/5. Remember that dividing by 9/5 is the same as multiplying by 5/9 and when we do this we end up with a slope of 5/3 at the given point.

20. If f of theta equals 4 sine theta minus theta at the origin, we can find the slope of the graph by first finding the derivative of 4 cosine theta minus one and then evaluating at theta equals zero. The slope of the graph of 4 sine theta minus theta at the origin is 3.
21. If it seems like all these examples are the same, they are for the most part. Let's make it more challenging. Let f of x equal x cubed minus 6 all over x squared. We want to find the derivative. Our steps, the same as always, will be to rewrite, take the derivative, then rewrite again. We still don't have a rule that allows us to deal with a variable in the denominator so in rewriting we will actually be simplifying. The denominator of x squared goes to each term of the numerator and then simplifying using exponent rules we have x minus 6 x to the negative 2. We can find the derivative which is 1 plus 12 x to the negative 3 and then rewrite with positive exponents again to get the derivative of f at x is equal to 1 plus 12 over x cubed.
22. Let y equal 3x multiplied by the quantity 6x minus 5 x squared. This is a product and although we have a sum and difference rule, we do not yet have a product rule. In rewriting we will distribute the 3 x to get y equals 18 x squared minus 15 x to the third power. This is a polynomial and has a straight forward derivative of 36 x minus 45 x squared.
23. For y equals 2 over the cubed root of x plus 3 cosine of x we rewrite the radical as a negative exponent, take the derivative remembering to subtract one from the exponent and finally rewrite to get y prime equals negative 2 over the quantity of 3 x to the four thirds minus 3 sine of x.
24. Determine the point, or possibly points, at which the graph of f of x equals x plus sine of x on the interval from 0 to 2 pi has a horizontal tangent line. Think about what we are being asked to do and/or find. A horizontal line has a slope of zero. A slope of a tangent line is a derivative. We want to find where the derivative is zero for the given function on the given interval. First we find the derivative of each term to get f prime of x equals 1 plus cosine of x. Next we set this derivative equal to zero and solve the trigonometric equation where cosine of x equals negative 1. For values of 0 and plus or minus 1, I think of the graph of cosine. I know based on the graph that the cosine function is negative one when x is pi.
25. Let's go back to the question/problem. Determine the points at which the graph has a horizontal tangent line. We have found the x values where the derivative is zero but we are being asked for points. Anytime we are asked for points on a graph we use our x values and go back to the original function to find y. Evaluating x plus sine of x at pi we get the y value of pi. That is, the graph of f of x equals x plus sine of x has a horizontal tangent line at the point pi comma pi in the given interval.
26. Our last example looks at the meanings of derivatives. Keep in mind that a derivative is a difference quotient, it is a generalized slope. The units of the derivative are the units of the

output of the function divided by the units of the input of the function. Suppose the number of gallons of regular unleaded gas sold by a gas station at a price of p dollars per gallon is given by N equals f of p. Our input is price in dollars per gallon and our output is N, the number of gallons sold. Problem one asks us to describe the meaning of f'(2.979). We know that this is the derivative at a price of 2.979 dollars per gallon. The derivative is number of gallons divided by dollars per gallon. This derivative is the rate of change in the number of gallons sold divided by price per gallon at 2.979 dollars per gallon. The second question is not mathematical as much as it is a real-life concern. Is this rate of change generally positive or negative? What do you think? When the price of gas is 2.797 per gallon, do you think the rate at which they sell gas is growing (positive) or falling (negative). I would say negative as stations will sell less fuel at higher prices.

27. This is the end of the lecture on basic derivative rules. Thanks to math is fun dot com for the graph of the constant function at the beginning.