Product and Quotient Rules and Higher-Order Derivatives

1. Welcome to product and quotient rules and higher-order derivatives. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include multiplying polynomials, rewriting radicals as rational exponents, simplifying rational expressions, exponent rules, and a firm grasp on the derivatives of sine and cosine.
3. Let's get started with Calculus I Derivatives: Product and Quotient Rules and Higher-Order Derivatives. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 2.3.
4. The product rule will allow us to find the derivative of functions that are written as products. It states, quite simply, that the product of two differentiable functions u and v is itself differentiable. But there is more; it also tells us how to find this derivative. If we want to find the derivative of the product u times v then the derivative is u multiplied by the derivative of v plus v multiplied by the derivative of u. If we call u the first function and v the second, you can remember this easier as "first times derivative of second plus second times derivative of first."
5. The more organized you are in your mathematics, the better your mathematics will be. For the given product quantity 6x plus 5 multiplied by quantity x cubed minus 2, let's identify u as the first function 6x plus 5 and v as the second function x cubed minus 2. Then we can find the derivative of u to be 6 and the derivative of v to be 3 x squared. Having all the pieces ready to use, we can now use the product rule and input the first (6x +5) times derivative of second (3x squared) plus second (x cubed minus 2) times derivative of the first (6). I encourage you to multiply and simplify when you can. It is not always required, but this is an algebra skill that you would be wise to brush up on if needed. These give you the perfect opportunity for practice.
6. Let's take a quick aside and see if we could find this derivative another way. If we were to multiply out first, like when we only had the power, constant and sum and difference rules, we could simplify f to a polynomial. Then using our previous rules we could find the derivative. Notice that this is the same derivative we found after using the product rule and then simplifying. Your choice is to simplify then take the derivative, or take the derivative and then simplify. Either way, you will be doing extensive algebra in simplifying with calculus.
7. Let's return to the product rule as we look at g of x equals square root of x multiplied by sine of x. As you read the function out loud, you can better tell which is the first function, square root of x, and which is the second, sine of x, by where you use the word "multiply." Rewriting the square root as a rational exponent of one half, we identify our u of x and find its derivative, then identify our v of x and find its derivative. Using these four pieces, we now put them together in

the product rule to get g prime of x equals x to the ½ times cosine of x plus sine of x times ½ x to the negative ½. As a final step we rewrite in the form of the original problem; it started with radicals so we finish with radicals.

8. Let h of x equal the quantity x to the negative two plus 1 over x multiplied by the quantity cube root of x minus cosine of x. First, I rewrite everything for use of the power rule. Next, identify the first function, the first set of parentheses, and find its derivative. Then find the second function, the second quantity, and find its derivative. Now use these four pieces appropriately in the product rule to arrive at the derivative given. Notice that no rewriting was done at the end of this problem as the original had negative exponents, we can leave the answer with negative exponents.
9. The product rule is given for a product of two functions. But what happens when you have more than two functions multiplied and you want to find a derivative? You can use a nested process wherein you use two functions as your "second" function and then use the product rule again on them. In doing so, you will find that the generalized product rule has all but one "pieces" of the product left alone multiplied by the derivative of the other piece. So with three functions multiplied you would leave two alone and multiply by the derivative of the third. You would have three terms in your addition as each piece would have to have a derivative taken.
10. For the function given by y equals quantity x squared plus 3x times quantity 2x minus 1 times quantity x to the fifth minus sine of x we leave the first two and find the derivative of x to the fifth minus sine of x. Add to that the first and last taking the derivative of 2x minus 1, and finally leave the last two and take the derivative of x squared plus 3x.
11. Since we have a product rule, it makes sense that we also have a quotient rule. (Though this is not true in all parts of Calculus.) If you are given the quotient of two differentiable functions, then it is also differentiable. Great, we know we can find the derivative, but what is it? In words, we take the denominator and multiply by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator and divide this difference by the original denominator squared. That seems confusing, but looking at a formula should help.
12. In symbols, the derivative of u over v is v times derivative of u minus u times derivative of v all over v squared, as long as v is not the zero function. Alternatively, you can think of the numerator as hi and denominator as lo and remember this rule as "lo de hi minus hi de lo over lo squared." The order matters since we are using subtraction!
13. Using the quotient rule for g of t equals t squared plus 4 all over 5 t minus 3 we identify our numerator as u of t equals t squared plus 4 and find its derivative. Then identify our denominator as v of t equals 5 t minus 3 and find its derivative. Now use these four pieces in the quotient rule to get g prime of t equals quantity 5 t minus 3 times 2t minus quantity t squared plus 4 times 5 all over 5t minus 3 squared.
14. Let's simplify this derivative. We do the individual multiplications in the numerator, being sure to distribute the negative sign from the subtraction. Combine like terms and then walk away. We could multiply the denominator, but this is rarely helpful so it is common to leave it as a square term.
15. For h of x equals x over quantity square root of x minus 1, the derivative is lo (square root of x minus 1) de hi (derivative of x is 1) minus hi (x) de lo (derivative of lo is ½ x to the negative ½) all

over lo squared. The bottom line contains the simplification: Multiplication by 1 leaves our first two terms the same, subtracting the rewritten x over 2 square root of x and leaving the denominator the same. This last equal sign contains some algebra that might not be immediately apparent; simplifying x over 2 square root of x by rationalizing the denominator leaves us with one half a square root. The square root of x minus half a square root of x leads us to the final version at the bottom of the page.

16. Using the quotient rule on f of x equals sine of x over x cubed, we have lo, x cubed, times de hi, cosine of x, minus hi, sine of x, de lo, 3 x squared, all over lo, x cubed, squared. This simplification is really just a rewriting without so many parentheses and simplifying the denominator appropriately.
17. When the quotient gets a little more complicated, it is a good idea to go back to identifying the numerator as u of x, finding its derivative, then the denominator as v of x and finding its derivative. Doing these more complicated derivatives first, allow us to keep our sanity when putting the pieces together in the quotient rule. Notice that the original function had negative exponents so I did not bother simplifying the derivative, leaving negative exponents.
18. Next, let's use both the product and quotient rules in the same problem. Let f of x equal the quantity x to the 7^{th} plus 3 x to the fourth be multiplied by the quotient sine of x over x. Using the product rule, we will have first (the polynomial) multiplied by the derivative of the quotient plus the quotient multiplied by the derivative of the polynomial. Notice I just used symbols to tell me what to do here, I have not taken any derivatives at this point. For the next line we take our individual derivatives, using u and v if needed for the quotient, and the power rule for the derivative of the denominator. In our final step we simplify by distributing appropriately and combining like terms.
19. As you can see, some of these problems get long. It is problems like this in which your organization is going to be extremely beneficial. Looking at this function y, the quotient is the overriding operation, as opposed to the previous problem where the product was the main operation. Because of this quotient, we start out with the quotient rule. Again, I write the terms in the rule with the appropriate derivative notation but not actually taking the derivatives yet. I'm forming a guide to help me substitute derivatives in the appropriate places. Notice within this quotient rule, I will use u and v in order to find the derivative of a product. I tried to find any part of me that wanted to simplify this and wasn't able to, so we will just leave it where it is.
20. In a previous lecture we used graphs to find the derivatives of sine and cosine. Keep in mind, the other four trig functions can be defined as quotient of these two basic functions. Now that we have a quotient rule, we can define the derivatives of the other four trigonometric functions. Notice that all the co-functions have negative derivatives. I'll go through the proof of one and leave it to you to prove the other three. Keep in mind, you don't have to memorize these as long as you always know the derivatives of sine and cosine along with the quotient rule.
21. Let's prove that the derivative of tangent of x is secant squared of x. First, let y equal tangent of x which is sine of x divided by cosine of x. Using the quotient rule we have lo (cosine) de hi (derivative of sine is cosine) minus hi (sine) de lo (derivative of cosine is negative sine) all over lo (cosine) squared. Though I do shortcut the language and say "sine" rather than "sine of x" and so on, it is important to remember these are trigonometric functions and without the variable

they have no meaning. Simplifying our numerator we get cosine squared x plus sine squared x over cosine squared x. But we know that cosine squared x plus sine squared x equals 1 as this is a Pythagorean identity. We also know that 1 over cosine of x is secant, so 1 over cosine squared x is secant squared x and we have proven the derivative.

22. Let's continue with the product and quotient rules now involving the other trig functions. Number 1 has f of theta equal to quantity theta plus one multiplied by cosine of theta. Using the product rule, f prime of theta is theta plus 1 multiplied by the derivative of cosine theta plus cosine theta multiplied by the derivative of theta plus 1. Take the second or two to do the simplification leaving this derivative as f prime of theta equals cosine of theta minus sine of theta minus theta times sine of theta. Number two is y equals x plus cotangent of x. Do not overthink this. The derivative of y is the derivative of x (which is 1) plus the derivative of cotangent of x (which is negative cosecant squared of x). Leave it and walk away.
23. For number three, this is a quotient function and so the derivative will use the quotient rule. The derivative of y is x times the derivative of secant of x (which is secant x tangent x) minus secant x times the derivative of x (which is one), all over x squared. Though we could leave this derivative, there are many different ways we could simplify it as well. In the bottom line I factored a secant out of the numerator. This might be useful when we have to solve in the future.
24. In number four, we use the product rule to find the derivative of x times sine of x and then just find the derivative of cosecant of x. But in number five we use the product rule twice, one for each term in the function. We take five theta and multiply by the derivative of secant theta then add secant theta times the derivative of five theta. Add to that theta multiplied by the derivative of tangent of theta plus tangent of theta times the derivative of theta. It is always best to clean up your derivatives and simplify to express your answer.
25. When finding the equation, or equations, of the tangent lines to a given graph it is important to keep in mind that the derivative is the slope. In this problem we have a further constraint that our slope must be parallel to a given line. Step 1, find the slope of the given line by solving for y. The slope of the given line is -1/2, so we want to find where the derivative is -1/2 in order to have tangent lines parallel to the given line.
26. In order to find where the derivative is equal to -1/2, we must first find the derivative using the quotient rule, and then solve.
27. In solving a proportion we can cross multiply and set the products equal to each other. Using the square root property, we know that x minus 1 is equal to plus OR minus 2. This will leave us with two x values where the derivative is equal to -1/2. All we need now to find the equations of the tangent lines are the y values so we can put it all together.
28. The derivative helps us find the appropriate x values, but we always use the original function to find the corresponding y values. Substituting -1 for x we find that y = 0 and then use the point-slope form of a line to find the equation of the tangent line through (-1,0). Substituting 3 for x we find that y =2. Once again the point-slope form of a line leads us to the equation of the tangent line through the point (3,2).
29. A derivative is not a one-time operation. Derivatives have derivatives and they have derivatives also. If we start with a position function, the derivative is velocity. The derivative of velocity is

acceleration. And the derivative of acceleration is known as the jerk, the speed at which you accelerate. Past that, we can still find derivatives but we don't have a real-life term or application that I'm aware of at this time. We could continue to use tick marks for derivatives, but after three it gets a little awkward. We use superscripts in parentheses to indicate higher order derivatives.

30. In this example, we are given a velocity function and are asked about the acceleration. The acceleration is the derivative of velocity so our first step is to find the derivative.
31. Using the quotient rule we have 2t minus 15 multiplied by the derivative of 100t minus 100t minus the derivative of 2t + 15 all divided by the denominator squared. I encourage you to simplify the numerator so it is easier to work with. Any time you will need to use the derivative you should first simplify it.
32. Now that we have the acceleration function we can evaluate at 5 seconds to get an acceleration of 2.4 feet per second squared.
33. At 10 seconds the acceleration is 1.22 feet per second squared.
34. At 20 seconds the acceleration is 0.50 feet per second squared. Notice that we accelerate at a higher rate early on but the acceleration drops off as time goes on and you get to your desired velocity.
35. The end of product and quotient rules and higher order derivatives.