The Chain Rule

1. Welcome to the Chain Rule. My name is Tuesday Johnson and I'm a lecturer at the University of Texas El Paso.
2. With each lecture I present, I will start you off with a list of skills for the topic at hand. You can find most of these reviews on my website, but if that doesn't work for you, you can find them pretty much anywhere in the internet world. My favorite places to look are Khan Academy and Math is Power 4 U. The skills for this lecture include multiplying polynomials, rewriting radicals as rational exponents, simplifying rational expressions, exponent rules, and a firm grasp on the derivatives of sine and cosine.
3. Let's get started with Calculus I Derivatives: The Chain Rule. This lecture corresponds to Larson's Calculus, 10^{th} edition, section 2.4.
4. In order to understand the chain rule, you must fully understand composition of functions. More importantly, you must understand how to decompose functions. When considering y = f of g of x I will refer to f as the outside function and g as the inside function. This terminology is used throughout the lecture.
5. The chain rule states that in order to find the derivative of a composition of functions, you should take the derivative of the outside function, leaving the inside function alone, and then multiplying by the derivative of the inside function.
6. The chain rule allows us to now define a general power rule that we will use frequently.
7. In finding the derivative of y equals 2 times the quantity 6 minus x squared all to the fifth power, we can identify the inside function as 6 minus x squared and the outside function as 2 times u to the fifth. These individual pieces are easy to find derivatives for and so we use them in the chain rule. The derivative of the function will be 10 u to the fourth times negative 2x. Then we replace the inside function u and finally simplify the derivative.
8. For the function g of x given by the square root of the quadratic x squared minus 2x plus 1 we identify the inside function as the quadratic and the outside function as the square root. We find each individual derivative and then put them together using the chain rule. Notice that in the numerator a 2 can be factored out in order to cancel the 2 already in the denominator.
9. The derivative of negative 5 over t plus 3 all cubed can be found in a couple of ways. I have chosen to take t plus 3 as the inside and negative five over u cubed as the outside. Rather than using the quotient rule, I have rewritten this quotient with a negative exponent. Putting the pieces together using the chain rule we have a derivative of 15 over the quantity t plus 3 to the fourth power.
10. Looking at y equals x over the square root of quantity x to the fourth plus 4 we see that we have a quotient and so we apply the quotient rule to find the derivative. We start with lo, the square root of x to the fourth plus 4 and multiply by the derivative of x. Then subtract hi, the x, and multiply by the derivative of the radical. This is where the chain rule will be used. Before we move on to that, don't forget to divide by the denominator squared.

11. Focusing just on the denominator the inside is x to the fourth plus four and the outside is the square root. We can find the derivative of both pieces and put them together to find the derivative to be 2 x cubed divided by the original radical.
12. Substituting this into our previous quotient rule and simplifying a couple of other items we arrive at an answer that could still be simplified, but we will leave it anyway. My rule is, if you need to use it you must simplify. However if you are just practicing derivatives and there is no further use, you do not have to simplify.
13. To find the derivative of a quotient raised to the second power, we identify the quotient as the inside and squared as the outside. While taking the derivative of the inside be sure to use the quotient rule. Simplification of this one is a little easier so it would be in your best interest to do so.
14. A nested function is one in which you will repeatedly use a rule working from the outside in. As I look at this function f, I see "stuff" to the third power. The derivative of that is 3 times the "stuff" all to the second power multiplied by the derivative of "stuff." Now I can focus on what the "stuff" is. The derivative of 2 plus something to the fourth power is 0 plus 4 times something all to the third power, multiplied by the derivative of "something." To finish we find the derivative of the something. You'll be asked to do this in the homework with next trig functions and radicals; take it slow and write it step by step to avoid confusion.
15. Moving on to trigonometric functions, let's find the derivative of sine of pi*x. Using parentheses appropriately can help you determine the inside from the outside so in this function, the sine is the outside and the inside is pi*x. This leads to a derivative of cosine, leave the inside alone, then multiply by the derivative of the inside. We will frequently rewrite this with the pi as a coefficient so that we don't accidentally multiply it inside the cosine function.
16. Let's let y equal the cosine of 1 minus 2x all squared. Here the argument of the trig function, the 1 – 2x is squared, not the cosine itself. We will work with two "insides" here so let v be the most inner function 1 – 2x. Then u is the "squared" part and finally y is the cosine. Take the derivative of the outermost function to get negative sine, leave the inside, and multiply by the derivative of the inside. In the next step we replace our u's and actually find the derivative of u where v is the variable. In the last calculus step we finally replace all v's and multiply by the derivative of v with x as the variable. The bottom line shows the algebraic simplification to make this derivative much easier to look at and use.
17. Over and over, the chain rule has us peeling a function apart like the layers of an onion. But as we use these high powered rules, don't forget the simple ones. The derivative of 3x is just 3. The coefficient of 5 will also come along for the ride. Now we slowly use the chain rule: the derivative of cosine is negative sine, leave the inside alone, then multiply by the derivative of the inside. The second math line finds the derivative of pi*x squared as 2*pi*x and then multiply by the derivative of the inside, the pi*x. Remember pi is just a constant so the third line shows the one change in the derivative of the inner most inside. Finish up with combining like terms and writing it in a more understandable notation.
18. Finding the derivative of y equal 3x minus 5 times the cosine of quantity pi*x squared. It is important to note the pi times x is squared and not the cosine. The "easy" part is the derivative of 3x which is 3. Moving on to the second term we keep the subtraction and the coefficient of 5.

Now to find the derivative of the cosine function. The derivative of cosine is negative sine, leave the inside alone, and then multiply by the derivative of the inside. In the next line for the derivative we start taking the derivative of pi times x all squared by bringing down the 2 and leaving the inside, pi*x, alone and multiplying by the derivative of the inside, pi*x. Our third line only takes the derivative of pi*x which is pi. Our final step is to simplify signs and multiply our coefficients all in the front.

19. In order to find the equation of the tangent line to a graph, we must first have a point and the slope of the tangent line at that point. Recall, the slope of the tangent line is always the derivative. For y equals quantity 4 times x cubed plus 3 all squared, we will look at the point -1 comma 1. Finding the derivative we take the derivative of the outside, "squared", leaving the inside alone, and then multiplying by the derivative of the inside. I encourage you to simplify this derivative since we will need to evaluate it. At the point where x = -1 the derivative is -24.
20. With the slope and point we can now use the point-slope form for the equation of a line to find the equation of the tangent line.
21. In our second example we are going to use a trigonometric function tangent squared of x at pi over 4 comma 1. In taking the derivative of tangent squared we remember that the squared is outside the tangent of x so the derivative is 2 times tangent of x times the secant squared of x. Evaluating this derivative we get a slope of 4.
22. With a slope of 4 and a point of pi over 4 comma 1, we use the point-slope form to find the equation of the tangent line is y equals 4x minus pi plus 1. You can always graph the original function and the tangent line in order to make sure you are correct.
23. Desmos dot com, the best free graphing site I have found online.