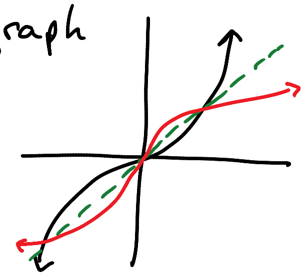
1.9 Inverse Functions

Definition: Let *f* and *g* be two functions such that $f\left(g\left(x\right)\right)=x$ for every *x* in the domain of *g* and $g\left(f\left(x\right)\right)=x$ for every *x* in the domain of *f*. Under these conditions, the function *g* is the inverse function of the function *f*. The function *g* is denoted by f^{-1} (read *f*-inverse). So, $f\left(f^{-1}\left(x\right)\right)=x$ and $f^{-1}\left(f\left(x\right)\right)=x$. The domain of *f* must equal the range of *f* inverse, and the range of *f* must be equal to the domain of *f* inverse.

Informally, an inverse function "undoes" what the original function did. For example, if $f\left(x\right)=x-6$ the inverse must be $f^{-1}\left(x\right)=x+6$.

Fact: The graph of a function and its inverse are symmetric across the line $y=x$.

If I'm given a graph I can draw in $y=x$ and then switch the points to get the graph of the inverse.

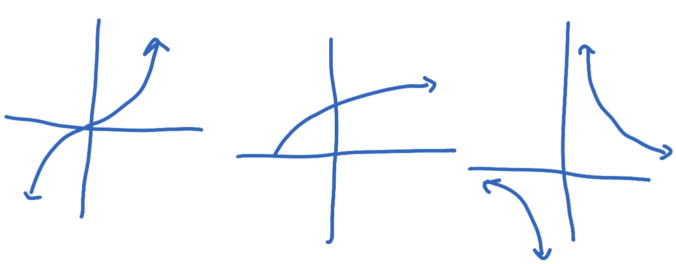


If a point (a,b) is on the graph of f, then the point (b,a) is on the graph of f^{-1}!

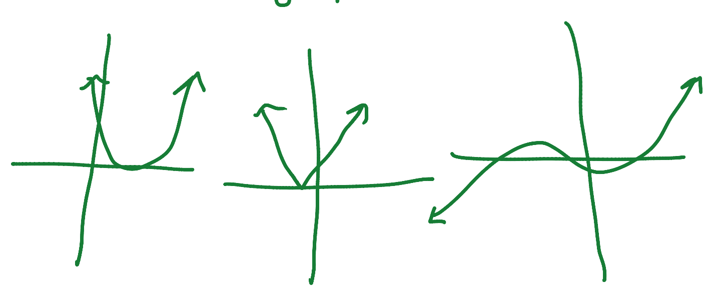
Horizontal Line Test for Inverse Functions – A function *f* has an inverse function if and only if no horizontal line intersects the graph of *f* at more than one point.

One-to-One Functions – A function *f* is one-to-one if each value of the dependent variable corresponds to exactly one value of the independent variable. A function *f* has an inverse function if and only if it is one-to-one.

1-1 Graphs



Not 1-1 graphs



Finding an Inverse Function:

1. Use the Horizontal Line Test to decide whether *f* has an inverse function.

2. In the equation for *f(x)*, replace *f(x)* by *y*.

3. Interchange the roles of *x* and *y*, and solve for *y*.

4. Replace *y* by $f^{-1}(x)$ in the new equation.

5. Verify that you have found the inverse using composition.

Examples: Find the inverse informally

1. $f(x)=\frac{1}{3}x$

The "opposite" of multiply by $\frac{1}{3}$ is dividing by $1/3$ so $f^{-1}(x)=\frac{x}{\frac{1}{3}}=3x$.

2. $f(x)=\frac{x-1}{5}$

start with x
Subtract 1
Divide by 5

inverse reverses order and operation

x
multiply by 5
add 1

$f^{-1}(x)=5x+1$

3. $f(x)=x^5$

The opposite of the 5^{th} power is the 5^{th} root.

$f^{-1}(x)=\sqrt[5]{x}$ or $f^{-1}(x)=x^{1/5}$

Example: Verify that $f(x)=\frac{x-9}{4}$ and $g(x)=4x+9$ are inverse functions.

Show that $f(g(x))=x$ and $g(f(x))=x$.

$f(g(x))=f(4x+9)=\frac{(4x+9)-9}{4}=\frac{4x}{4}=x$ ✓

$g(f(x))=g\left(\frac{x-9}{4}\right)=4\left(\frac{x-9}{4}\right)+9=(x-9)+9=x$ ✓

Example: Verify that $f(x) = x^3 + 5$ and $g(x) = \sqrt[3]{x-5}$ are inverse functions.

$f(g(x)) = f(\sqrt[3]{x-5}) = (\sqrt[3]{x-5})^3 + 5 = x - 5 + 5 = x$

$g(f(x)) = g(x^3+5) = \sqrt[3]{(x^3+5)-5} = \sqrt[3]{x^3} = x$

Example: Find the inverse function.

1. $f(x) = 3x + 5$

$y = 3x + 5$

$x = 3y + 5$

$x - 5 = 3y$

$\frac{x-5}{3} = y$

$f^{-1}(x) = \frac{x-5}{3}$

Check: f: mult 3, add 5; f^{-1}: sub 5, divide by 3

2. $f(x) = \frac{3x+4}{5}$

$y = \frac{3x+4}{5}$

$5 \cdot x = \frac{3y+4}{\cancel{5}} \cdot \cancel{5}$

$5x = 3y + 4$

$5x - 4 = 3y$

$\frac{5x-4}{3} = y$

$f^{-1}(x) = \frac{5x-4}{3}$

Check: f: m 3, a 4, d 5; f^{-1}: m 5, s 4, d 3

3. $f(x) = \sqrt{x-2}$

$y = \sqrt{x-2}$

$x = \sqrt{y-2}$

$x^2 = (\sqrt{y-2})^2$

$x^2 = y - 2$

$x^2 + 2 = y$

$f^{-1}(x) = x^2 + 2, \; x \geq 0$

Check: f: sub 2, sq root; f^{-1}: square, add 2

for f, domain is $x \geq 2$ and range is $y \geq 0$ so we must restrict the domain of $f^{-1}(x)$ to be $x \geq 0$

4. $g(x)=\dfrac{3x-1}{2x+5}$

$$y=\frac{3x-1}{2x+5}$$

$$(2y+5)\,x=\frac{3y-1}{\cancel{2y+5}}\,(\cancel{2y+5})$$

$$(2y+5)x=3y-1$$

$$2xy+5x=3y-1$$
$$-3y \quad -5x \quad -3y \quad -5x$$
$$2xy-3y=-1-5x$$
$$y(2x-3)=-1-5x$$
$$y=\frac{-1-5x}{2x-3}$$

$$g^{-1}(x)=\frac{-(1+5x)}{2x-3}$$

or

$$g^{-1}(x)=-\frac{1+5x}{2x-3}$$

or

$$g^{-1}(x)=\frac{1+5x}{3-2x}$$

5. $f(x)=\dfrac{x+7}{3-5x}$

$$y=\frac{x+7}{3\quad 5x}$$

$$(3-5y)x=\frac{y+7}{\cancel{3-5y}}\,(\cancel{3-5y})$$

$$3x-5yx=y+7$$
$$3x-7=y+5yx$$
$$3x-7=(1+5x)y$$

$$\frac{3x-7}{1+5x}=y$$

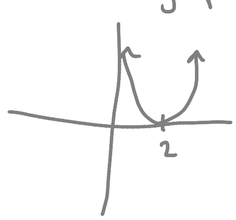
$$f^{-1}(x)=\frac{3x-7}{1+5x}$$

Examples: Restrict the domain so that the function has an inverse then find that inverse.

1. $f(x)=(x-2)^2$

Quick graph

right 2



This graph is not 1–1 so the function does not have an inverse. However, if we chose half of it we're good. Choose either part, for this example we use $f(x)=(x-2)^2,\ x\geq 2$.

The inverse: $y=(x-2)^2$

$$x=(y-2)^2$$
$$\pm\sqrt{x}=y-2$$
$$2\pm\sqrt{x}=y$$

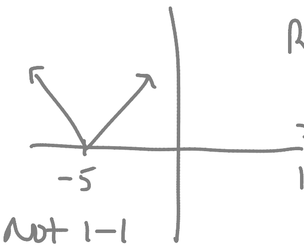
But which is it? plus or minus?

f had $x\geq 2$ so f^{-1} must have $y\geq 2$

This leads to the decision that $y=2+\sqrt{x}$ or $f^{-1}(x)=2+\sqrt{x}$

2. $f(x) = |x+5|$

Quick Graph



Not 1-1

Restrict $x \geq -5$ to make it easy

When $x \geq -5$,
$f(x) = |x+5|$
behaves as
$y = x+5$

$x = y+5$
$x-5 = y$
$f^{-1}(x) = x-5$
for restricted domain.

Example: Your wage is \$10.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage *y* in terms of the number of units produced is *x* is $y = 10 + 0.75x$.

a) Find the inverse function. What does each variable represent in the inverse function?

$x = 10 + 0.75y$
$x - 10 = 0.75y$
$\frac{x-10}{0.75} = y$

$\frac{4}{3}x - \frac{40}{3} = y^{-1}$

x ↑ Now wage

y^{-1} ↑ Now units produced

b) Determine the number of units produced when your hourly wage is \$24.25.

$y^{-1} = \frac{4}{3}(24.25) - \frac{40}{3} = 19$ units

Note: I do <u>not</u> recommend switching variables $x \leftrightarrow y$ when finding the inverse in a word problem. It becomes too easy to lose track of what the variables represent.