2.6 Rational Functions

Informal Definition of a Rational Function – A rational function is a quotient of polynomial functions. It can be written in the form $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

Domain of a Rational Function – In general, the domain of a rational function includes all real numbers except those that make the denominator zero.

Definitions of Vertical and Horizontal Asymptotes –

1. The line *x* = *a* is a vertical asymptote of the graph of *f* if $f(x)\to\infty \;\; or \;\; f(x)\to-\infty$ as $x\to a$, either from the right or from the left.

2. The line *y* = *b* is a horizontal asymptote of the graph of *f* if $f(x)\to b$ as $x\to\pm\infty$.

Important: Asymptotes are always equations of lines and never just a number.

$y=4$ is a horizontal asymptote; 4 is just a number

$x=-2$ is a vertical asymptote; -2 is not.

Vertical and Horizontal Asymptotes of a Rational Function – Let *f* be the rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

where *N(x)* and *D(x)* have no common factors.

1. The graph of *f* has vertical asymptotes at the zeros of *D(x)*.

2. The graph of *f* has one or no horizontal asymptote determined by comparing the degrees of *N(x)* and *D(x)*.

    a) If $n < m$, the graph of *f* has the line $y = 0$ (the *x*-axis) as a horizontal asymptote.

    b) If $n = m$, the graph of *f* has the line $y = \frac{a_n}{b_m}$ as a horizontal asymptote.

    c) If $n > m$, the graph of *f* has no horizontal asymptote.

Examples: Find the domain of the rational functions then identify any vertical and horizontal asymptotes.

1. $f(x) = \frac{4}{(x-2)^3}$

domain: $(x-2)^3 \neq 0$
$x-2 \neq 0$
$x \neq 2$

V.A. $X=2$

Closely related to domain

H.A. $y=0$, denom. has a larger degree.

2. $f(x) = \frac{3-7x}{3+2x}$

Domain: $3+2x \neq 0$
$2x \neq -3$
$x \neq -\frac{3}{2}$

V.A. $X = -\frac{3}{2}$

H.A. $y = -\frac{7}{2}$

degree of denom = degree of num.

3. $f(x) = \frac{4x^2}{x+2}$

Domain: $X+2 \neq 0$
$X \neq -2$

V.A.
$X = -2$

H.A. $y = \ldots$ $\boxed{\text{NONE}}$

deg denom < deg numerator so no horizontal asymptote

Guidelines for Analyzing Graphs of Rational Functions – Let $f(x)=\frac{N(x)}{D(x)}$, where *N(x)* and *D(x)* are polynomials.

1. Simplify *f*, if possible.

2. Find and plot the y-intercept (if any) by evaluating *f(0)*.

3. Find the zeros of the numerator (if any) by solving the equation *N(x) = 0*. Then plot the corresponding *x*-intercepts.

4. Find the zeros of the denominator (if any) by solving the equation *D(x) = 0*. Then sketch the corresponding vertical asymptotes.

5. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.

6. Plot at least one point between and one point beyond each *x*-intercept and vertical asymptote.

7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

Quick summary: 1 $D(x) \neq 0$ domain 2 $D(x)=0, N(x)\neq 0$ vert. asymptote
3 $D(x)=0, N(x)=0$ hole in graph 4 $D(x)\neq 0, N(x)=0$ x-intercept
5 $f(0)=$ y-intercept 6 degrees for horizontal asymptote

numerator denominator both numerator and denominator

Examples: Sketch the graph of the rational functions.

1. $f(x)=\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$ if we simplify

Domain: $x \neq -1, 1$

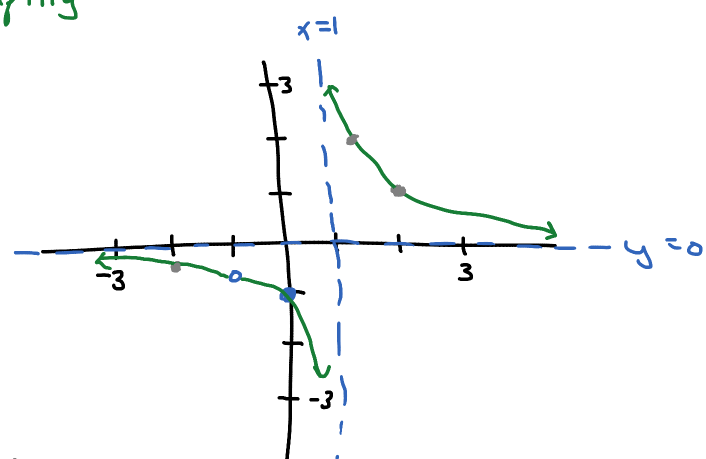
hole at $x=-1$

vert. asymptote $x=1$

no x-intercept

y-intercept $(0,-1)$

horizontal asymptote $y=0$



random points to fill in: $(-2,-\frac{1}{3})$
$(1.5, 2)$
$(2, 1)$

Basic rational graph

2. $f(x) = \frac{x^2-4}{x^2-3x+2} = \frac{(x+2)(x-2)}{(x-2)(x-1)} = \frac{x+2}{x-1}$

Domain: $x \neq 2, 1$

hole at $x=2$

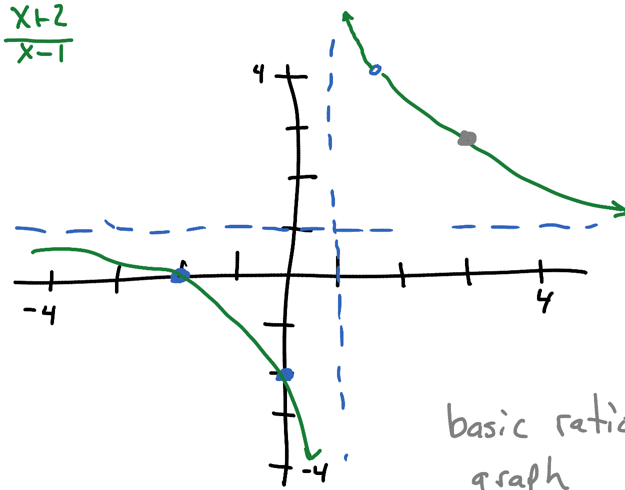
Vert. asy $x=1$

x-int at $x=-2$ $(-2,0)$

y-int at $y = -\frac{4}{2} = -2$ $(0,-2)$

horiz. asy $y = \frac{1}{1}$ or $y=1$

extra point $(3, 2.5)$



basic rational graph

3. $f(x) = \frac{1}{x-3}$

Domain: $x \neq 3$

V.A. $x=3$

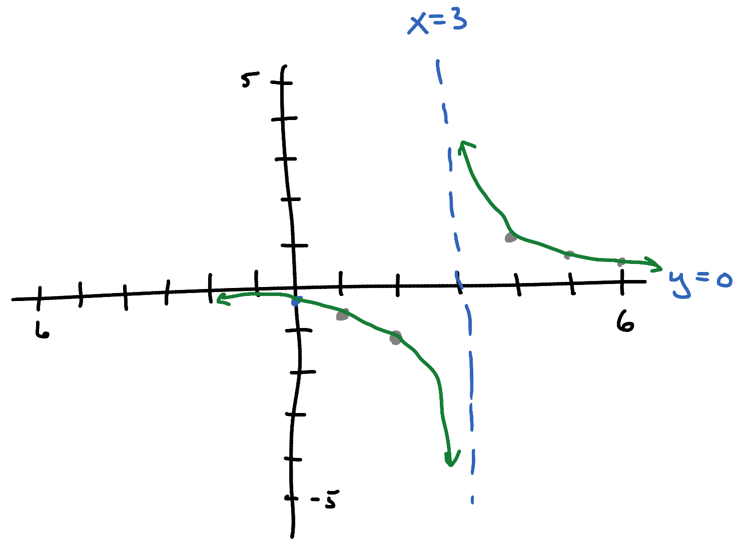
H.A. $y=0$

no x-int

y-int $(0, -\frac{1}{3})$

No holes

extra points: $(1, -\frac{1}{2})$ $(2,-1)$ $(4,1)$ $(5,\frac{1}{2})$ $(6,\frac{1}{3})$



Note: The graph of $f(x) = \frac{1}{x-3}$ is just the graph of $y = \frac{1}{x}$ shifted right 3 units.

4. $f(x)=\frac{1-2x}{x}$

points $(-1,-3)$ $(-2,-2.5)$
$(1,-1)$ $(2,-1.5)$

Domain: $x \neq 0$

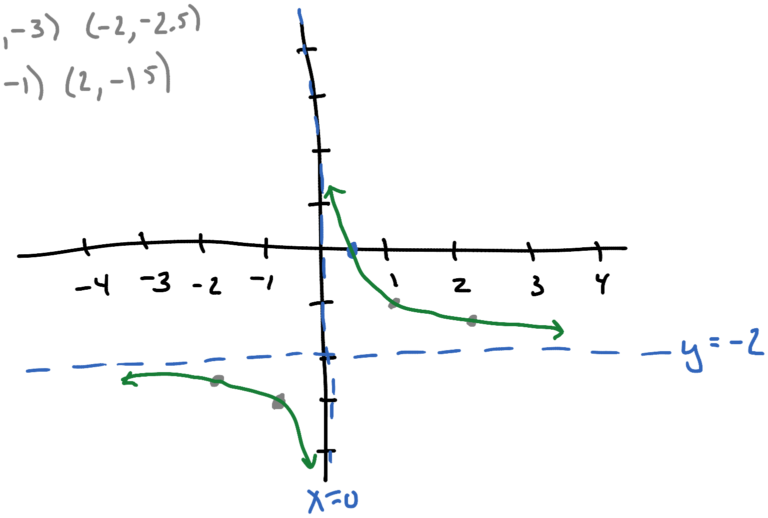
V.A. $x=0$

no holes

x-int $1-2x=0$
$x=\frac{1}{2}$ $(\frac{1}{2},0)$

no y-int

H.A. $y=-\frac{2}{1}$ or $y=-2$



5. $f(x)=\frac{2x^2-5x-3}{x^3-2x^2-x+2}=\frac{(2x+1)(x-3)}{(x-2)(x+1)(x-1)}$

Domain: $x \neq 2,-1,1$

V.A. $x=2, x=-1, x=1$

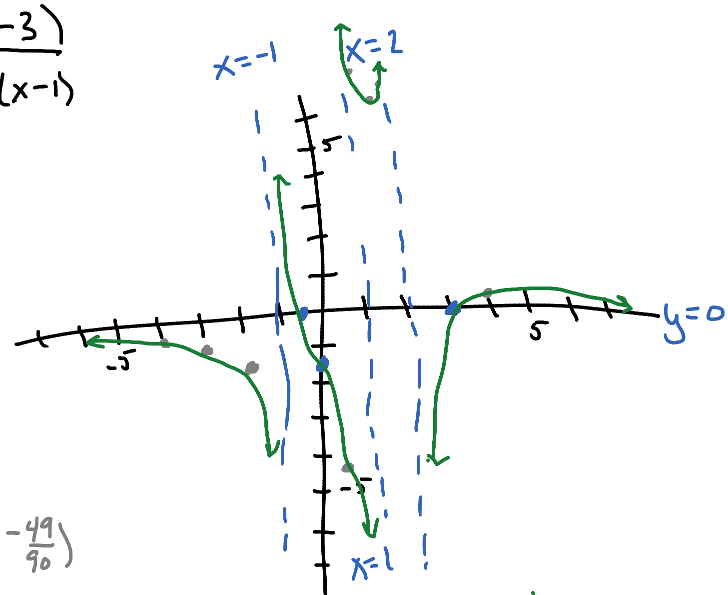
H.A. $y=0$

x-int $(-\frac{1}{2},0),(3,0)$

y-int $(0,-\frac{3}{2})$

no holes

extra points: $(-2,-\frac{5}{4})$ $(-3,-\frac{3}{4})$ $(-4,-\frac{49}{90})$
$(\frac{1}{2},-4.\overline{4})$
$(1.25,14.5)$ $(1.5,9.6)$ $(1.75,10.9)$
$(4,\frac{3}{10})$ $(5,\frac{11}{36})$



odd graph!!

But totally do-able.

If the degree of the numerator is exactly one more than the degree of the denominator, the graph of the function has a slant (or oblique) asymptote. To find the slant asymptote, divide and look for the quotient.

Examples: Find the slant asymptotes.

1. $f(x)=\dfrac{x^2+5}{x} = \dfrac{x^2}{x}+\dfrac{5}{x} = x+\dfrac{5}{x}$ ← ignore remainder

quotient $y=x$ is slant asymptote

2. $f(x)=\dfrac{x^2}{x-1}$

$$\begin{array}{c|ccc} & x^2 & x & \text{const} \\ 1 & 1 & 0 & 0 \\ & & 1 & 1 \\ \hline & 1 & 1 & 1 \\ & x & \text{const} & R \end{array}$$

$y=x+1$ is slant asymptote

3. $f(x)=\dfrac{2x^2-5x+5}{x-2}$

$$\begin{array}{c|ccc} 2 & 2 & -5 & 5 \\ & & 4 & -2 \\ \hline & 2 & -1 & 3 \end{array}$$

$y=2x-1$ is slant asymptote

always an equation not an expression.

That is, $y=2x-1$ is a slant asymptote, $2x-1$ is not!