3.4 Exponential and Logarithmic Equations

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the one-to-one properties of exponential and logarithmic functions.

2. Rewrite an exponential equation in logarithmic form and apply the inverse property of logarithmic functions.

3. Rewrite a logarithmic equation in exponential form and apply the inverse property of exponential functions.

Examples: Solve.

1. $4^x = 16$
   
   We can rewrite each side of this equation with the same base in order to use the one-to-one property of equality:

   $4^x = 4^2$ therefore $x = 2$

2. $\left(\frac{1}{4}\right)^x = 64$
   
   Rewriting each side we get $\frac{1}{4} = 4^{-1}$ and $64 = 4^3$

   Which becomes $(4^{-1})^x = 4^3$ or $4^{-x} = 4^3$. Therefore $-x = 3$ and $x = -3$.

3. $\ln x - \ln 5 = 0$
   
   The easiest way would be to add $\ln 5$ to both sides for the use of the one-to-one property of equality:

   $\ln x = \ln 5$ becomes $x = 5$.

4. $e^x = 5$
   
   There is no chance of rewriting to the same base so we convert forms: If $e^x = 5$, then $x = \ln 5 \approx 1.609$
5. \( \log x = -2 \)
   
   Switch forms with a base of 10
   
   \( 10^{-2} = x \)
   
   So \( \frac{1}{10^2} = x \) or \( 10^{-2} = x \)

6. \( \log_5 x = \frac{1}{2} \)
   
   \( 5^{1/2} = x \)
   
   \( \sqrt{5} = x \approx 2.236 \)

7. \( e^{2x} = e^{x-8} \)
   
   Same base so: \( 2x = x - 8 \)
   
   To solve a quadratic, set it equal to zero and factor or use the formula
   
   \( \Delta = x - 2x - 8 \)
   
   \( \Delta = (x-4)(x+2) \)
   
   \( x = 4 \) or \( x = -2 \)

8. \( 2(5^x) = 32 \)
   
   Isolate the exponential: \( \frac{2(5^x)}{2} = \frac{32}{2} \)
   
   \( 5^x = 16 \)
   
   Now rewrite to solve: \( x = \log_5(16) \approx 2.723 \)
   
   \( x \) is the power to which \( 5 \) is raised to get 16

9. \( 6^x + 10 = 47 \)
   
   Isolate: \( 6^x + 10 - 10 = 47 - 10 \)
   
   \( 6^x = 37 \)
   
   Rewrite: \( x = \log_6(37) \approx 2.015 \)

10. \( 4^{-3t} = 0.10 \)
   
   \( -3t = \log_4(0.10) \)
   
   \( t = \frac{\log_4(0.10)}{-3} \)
   
   \( t = \frac{\log(0.10)}{\log(4)} \approx 0.554 \)
11. $2^{x-3} = 32$
   
   Let it be easy when you can
   
   $2^{x-3} = 2^5$
   
   so $x - 3 = 5$
   
   $x = 8$

12. $8(3^{6-x}) = 40$
   
   $\frac{8}{3^{6-x}} = \frac{40}{8}$
   
   $3^{6-x} = 5$
   
   $-x = \log_3 5 - 6$
   
   $x = 6 - \log_3 5$
   
   $x \approx 4.535$

13. $e^{2x} - 5e^x + 6 = 0$
   
   This is called quadratic in form because it looks like $y^2 - 5y + 6 = 0$.
   
   For this reason we factor:
   
   $(e^x - 3)(e^x - 2) = 0$
   
   $e^x - 3 = 0$ or $e^x - 2 = 0$
   
   $e^x = 3$ or $e^x = 2$
   
   $x = \ln 3 \approx 1.099$ or $x = \ln 2 \approx 0.693$

14. $e^{2x} + 9e^x - 36 = 0$
   
   $(e^x + 12)(e^x - 3) = 0$
   
   $e^x + 12 = 0$ or $e^x - 3 = 0$
   
   $e^x = -12$ or $e^x = 3$
   
   Not in range of $e$
   
   $x = \ln 3 \approx 1.099$

15. $\ln(x+1) - \ln(x-2) = \ln x$
   
   Rewrite the left:
   
   $\ln \frac{x+1}{x-2} = \ln x$
   
   Use 1-1 property:
   
   $\frac{x+1}{x-2} = x$
   
   Solve:
   
   $(x-1) \cdot \frac{x+1}{x-2} = x \cdot (x-1)$
   
   $x + 1 = x^2 - 2x - x - 1$
   
   $0 = x^2 - 3x - 1$
   
   $x = \frac{3\pm \sqrt{13}}{2}$
   
   $x > \frac{3+\sqrt{13}}{2}$ or $x < \frac{3-\sqrt{13}}{2}$

   $x > \frac{3+\sqrt{13}}{2}$
   
   Bigger than $3$ so ok

   $x < \frac{3-\sqrt{13}}{2}$
   
   Negative so not ok
16. \( \log_{4} x - \log_{4} (x-1) = \frac{1}{2} \)

- **Combine:** \( \log_{4} \frac{x}{x-1} = \frac{1}{2} \)
- **Rewrite:** \( 4^{\frac{1}{2}} = \frac{x}{x-1} \)
- **Simplify:** \( \sqrt{4} = 2 = \frac{x}{x-1} \)

- **Solve:** \((x-1)2 = \frac{x}{x-1} \cdot (x-1)\)
  
  \( 2x - 2 = x \)
  
  \( x = 2 \)

17. \( \log_{3} x + \log_{3} (x-8) = 2 \)

- \( \log_{3} (x^{2} - 8x) = 2 \)
  
  \( \frac{2}{3} = x - 8x \)
  
  \( 9 = x - 8x \)
  
  \( 0 = x^{2} - 8x - 9 \)

- **Solve:** \( \Delta = (x-9)(x+1) \)
  
  \( x-9 = 0 \quad \text{or} \quad x+1 = 0 \)
  
  \( x = 9 \quad \text{or} \quad x = -1 \)

**Note:** Not in the domain of the original expression; so not an acceptable answer.