3.5 Exponential and Logarithmic Models

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows:

1. Exponential growth model
   \[ y = ae^{bx}, \ b > 0 \]

2. Exponential decay model
   \[ y = ae^{-bx}, \ b > 0 \]

3. Gaussian model
   \[ y = ae^{-(x-c)^2/c} \]

4. Logistic growth model
   \[ y = \frac{a}{1 + be^{-rx}} \]

5. Logarithmic models
   \[ y = a + b \ln x, \ y = a + b \log x \]

Examples:

1. Determine the principal P that must be invested at 5%, compounded monthly, so that $500,000 will be available for retirement in 10 years.

   \[ A = P \left(1 + \frac{0.05}{12}\right)^{12t} \]
   \[ 500,000 = P \left(1 + \frac{0.05}{12}\right)^{120} \]
   \[ 500,000 = P \left(1.004167\right)^{120} \]
   \[ \frac{500,000}{\left(1.004167\right)^{120}} = P = \$303580.52 \]

2. Determine the time necessary for $1000 to double if it is invested at 6.5% if it is compounded

   a) annually
   \[ A = 2000 \]
   \[ \frac{2000 = 1000 \left(1 + \frac{0.065}{1}\right)^{1t}}{1000} \]
   \[ t = 10 \log_{1.065} (2) \]
   \[ t = 11 \text{ years} \]

   b) monthly
   \[ 2000 = 1000 \left(1 + \frac{0.065}{12}\right)^{12t} \]
   \[ 2 = \left(1 + \frac{0.065}{12}\right)^{12t} \]
   \[ 12t = \log_{1.005417} (2) \]
   \[ t = \frac{\log_{1.005417} (2)}{12} \approx 10.69 \text{ yrs} \]

   c) daily
   \[ 2000 = 1000 \left(1 + \frac{0.065}{365}\right)^{365t} \]
   \[ 2 = \left(1 + \frac{0.065}{365}\right)^{365t} \]
   \[ 365t = \log_{1.000183} (2) \]
   \[ t = \frac{\log_{1.000183} (2)}{365} \approx 10.66 \text{ yrs} \]
3. Carbon 14 decays with a half-life of 5715 years. Find how much remains from a 6.5 g sample after 1000 years.

\[ y = a e^{-bx} \]

where \( a \) is initial amount, \( b \) is decay constant and \( x \) is time

**Half-life**: amount of time for \( \frac{1}{2} \) substance to decay.

**Strategy**: we know \( a = 6.5 \text{g} \) but we must find decay constant \( b \)

Then we use \( a + b \) to find \( y \):

\[
\frac{1}{2} = e^{-5715b} \implies -5715b = \ln(\frac{1}{2}) \implies b = \frac{-\ln(\frac{1}{2})}{-5715} = 1.213 \times 10^{-4}
\]

So \( b = 0.0001213 \)

Now \( y = 6.5 e^{-0.0001213(1000)} = 5.7576 \) or approx 5.8 grams

4. The populations \( P \) (in thousands) of Horry County, South Carolina from 1970 through 2007 can be modeled by \( P = 18.5 + 92.2e^{0.0282t} \), where \( t \) represents the year, with \( t = 0 \) corresponding to 1970.


<table>
<thead>
<tr>
<th>Years since 1970</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>37</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population in thousands</td>
<td>73.7</td>
<td>103.7</td>
<td>143.6</td>
<td>196.9</td>
<td>243.2</td>
</tr>
</tbody>
</table>

Not 73.7 people, but 73.7 thousand people = 73.7\times1000 = 73,700

b) According to the model, when will the population of Horry County reach 300,000?

\[ 300,000 = 300 \text{ thousand} \text{ so we solve } 300 = -18.5 + 92.2e^{0.0282t} \text{ for } t. \]

\[ t = \frac{\ln\left(\frac{318.5}{92.2}\right)}{0.0282} \]

\[ 318.5 = 92.2 e^{0.0282t} \]

\[ \ln\left(\frac{318.5}{92.2}\right) = 0.0282t \]

\[ 318.5 = 92.2 e^{0.0282t} \]

\[ t = 43.95 \text{ or 44 yrs from 1970} \rightarrow \text{In the year } 2014. \]
c) Do you think the model is valid for long-term predictions of the population?

What do you think and why?

5. The number of bacteria in a culture is increasing according to the law of exponential growth. After 3 hours, there are 100 bacteria, and after 5 hours, there are 400 bacteria. How many bacteria will there be after 6 hours?

We do not know initial amount or how quickly they grow, growth rate. But we do know two values:

3 hrs, 100 bacteria gives $100 = ae^{b(3)}$ or $100 = ae^{3b}$
5 hrs, 400 bacteria gives $400 = ae^{5b}$. The $a + b$ in each equation is the same so we solve one equation for $a$: $\frac{100}{e^{3b}} = a$ then substitute it into the other equation: $400 = \frac{100 e^{5b}}{e^{3b}}$. Now we have one equation with one variable and we can find the value of $b$.

$400 = \frac{100 e^{5b}}{e^{3b}} \rightarrow 400 = 100 e^{2b} \rightarrow y = e^{2b} \rightarrow 2b = \ln 4 \rightarrow b = \frac{\ln(4)}{2} \approx 0.6931$

With $b$, we can find $a$ in the blue equation: $\frac{100}{e^{3(0.6931)}} = a \approx 12.5$

But we’re not finished yet! The question asks how many bacteria after 6 hours. $y = ae^{bx}$ at 6 hours is $y = 12.5 e^{0.6931(6)} \approx 799.8$ or 800 bacteria.

You can also use a graphing calculator to find an exponential regression. Be careful! My calculator gave $a = 12.5$ and $b = 2$. You must pay attention to the form of the answer.*

We used $y = ae^{bx}$ to find $a$ and $b$. The calculator uses $y = a(b)^x$. Either way, $y = 12.5(2)^6 = 800$. 

*Note: The asterisk indicates a critical detail about the calculation process.
6. At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person’s temperature twice. At 9:00 A.M. the temperature was 85.7 degrees, and at 11:00 A.M. the temperature was 82.8 degrees. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula 

\[ t = -10 \ln \left( \frac{T - 70}{98.6 - 70} \right) \]

where \( t \) is the time in hours elapsed since the person died and \( T \) is the temperature (in degrees F) of the person’s body. (This formula is derived from Newton’s Law of Cooling.) Use the formula to estimate the time of death of the person.

\[ t = -10 \ln \left( \frac{85.7 - 70}{98.6 - 70} \right) = 6 \text{ hours earlier} \]

\[ t = -10 \ln \left( \frac{82.8 - 70}{98.6 - 70} \right) = 3 \text{ hours earlier} \]

either way, time of death was approximately 3:00 A.M.