4.3 Right Triangle Trigonometry

The second way that the six trig functions can be defined is through a right triangle.

Right Triangle Definitions of Trigonometric Functions: Let θ be an acute angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows.

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

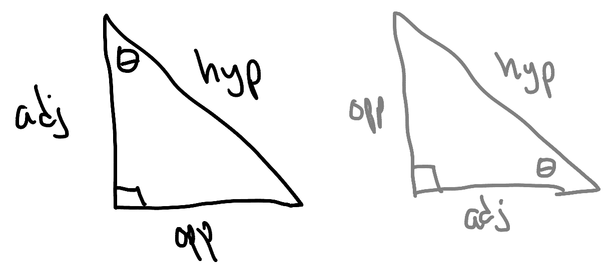
$$\csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.

opp = the length of the side opposite θ

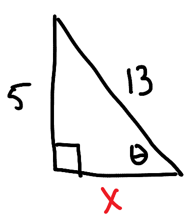
adj = the length of the side adjacent to θ

hyp = the length of the hypotenuse



Examples: Find the exact values of the six trig functions of the angle θ.

1.



$5^2 + x^2 = 13^2$

$25 + x^2 = 169$

$x^2 = 144$

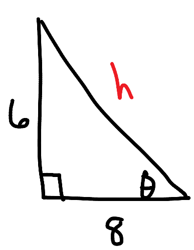
$x = 12$

$\sin\theta = \frac{5}{13}$ $\qquad \csc\theta = \frac{13}{5}$

$\cos\theta = \frac{12}{13}$ $\qquad \sec\theta = \frac{13}{12}$

$\tan\theta = \frac{5}{12}$ $\qquad \cot\theta = \frac{12}{5}$

2.



$6^2 + 8^2 = h^2$

$36 + 64 = h^2$

$100 = h^2$

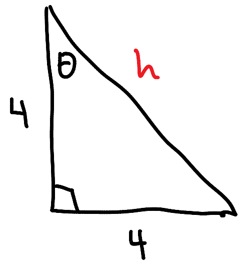
$10 = h$

$\sin\theta = \frac{6}{10} = \frac{3}{5}$ $\qquad \csc\theta = \frac{5}{3}$

$\cos\theta = \frac{8}{10} = \frac{4}{5}$ $\qquad \sec\theta = \frac{5}{4}$

$\tan\theta = \frac{6}{8} = \frac{3}{4}$ $\qquad \cot\theta = \frac{4}{3}$

3.



$4^2 + 4^2 = h^2$

$16 + 16 = h^2$

$32 = h^2$

$\sqrt{32} = h$

$4\sqrt{2} = h$

$\sin\theta = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\csc\theta = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\cos\theta = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\sec\theta = \sqrt{2}$

$\tan\theta = \frac{4}{4} = 1$ $\quad$ $\cot\theta = 1$

Sine, Cosine, and Tangents of Special Angles

Degrees	Radians	$\sin t$	$\cos t$	$\tan t$
30	π/6	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45	π/4	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$
60	π/3	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

Fundamental Trigonometric Identities

Reciprocal functions

$$\sin\theta = \frac{1}{\csc\theta} \quad \cos\theta = \frac{1}{\sec\theta} \quad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

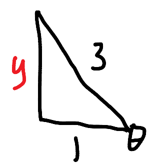
Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example: Given $\cos\theta = \frac{1}{3}$, find the values of $\sin\theta$, $\tan\theta$, and $\sec\theta$.

Cosine is $\frac{\text{adj}}{\text{hyp}}$ so



$3^2 = 1^2 + y^2$

$9 = 1 + y^2$

$8 = y^2$

$2\sqrt{2} = y$

$\sin\theta = \frac{2\sqrt{2}}{3}$

$\tan\theta = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$

$\sec\theta = \frac{1}{\cos\theta} = 3$

Example: Use the trig identities to transform the left side of the equation into the right side.

LHS = left hand side

RHS = right hand side

1. $\cos\theta\sec\theta = 1$

$LHS = \cos\theta\sec\theta = \cos\theta \frac{1}{\cos\theta} = \frac{\cos\theta}{\cos\theta} = 1 = RHS$

2. $\tan\alpha\cos\alpha = \sin\alpha$

$LHS = \tan\alpha\cos\alpha = \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\cos\alpha}{1} = \frac{\sin\alpha\cancel{\cos\alpha}}{\cancel{\cos\alpha}} = \sin\alpha = RHS$

3. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$LHS = \sec^2\theta - \cancel{\sec\theta\tan\theta} + \cancel{\tan\theta\sec\theta} - \tan^2\theta$

$= \sec^2\theta - \tan^2\theta$

$= 1 + \tan^2\theta - \tan^2\theta$

$= 1$

$= RHS$

4. $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$

The RHS still has $\sin\theta$, so use this as a clue to eliminate $\cos\theta$ from the left

$LHS = \sin^2\theta - (1 - \sin^2\theta)$

$= \sin^2\theta - 1 + \sin^2\theta$

$= 2\sin^2\theta - 1$

$= RHS$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

5. $\dfrac{\tan\beta + \cot\beta}{\tan\beta} = \csc^2\beta$

Common denominators are nice, but sometimes we like to split fractions. This is easy to do with a single term in the denominator

$$LHS = \frac{\tan\beta}{\tan\beta} + \frac{\cot\beta}{\tan\beta}$$

$$= 1 + \cot\beta \cdot \frac{1}{\tan\beta}$$

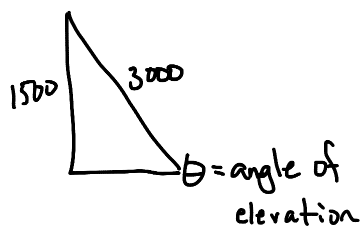
$$\cot\beta = \frac{1}{\tan\beta}$$

$$= 1 + \cot\beta \cdot \cot\beta$$

$$= 1 + \cot^2\beta$$

$$= \csc^2\beta = RHS$$

Example: You are skiing down a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?

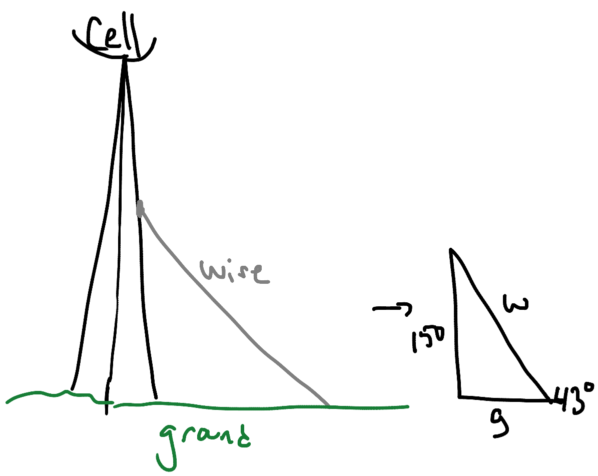


We have opp and hyp so it must be

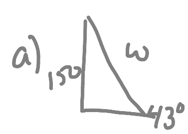
than $\sin\theta = \frac{1500}{3000} = \frac{1}{2}$. This is one I've asked you to memorize.

$\sin\theta = \frac{1}{2}$ when $\theta = 30^\circ = \frac{\pi}{6}$

Example: A guy wire runs from the ground to a cell tower. The wire is attached to the cell tower 150 feet above the ground. The angle formed between the wire and the ground is 43°. (a) How long is the guy wire? (b) How far from the base of the tower is the guy wire anchored to the ground?



a)



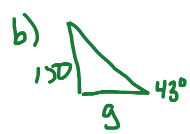
is $\sin 43^\circ = \frac{150}{w}$

$w \sin 43^\circ = 150$

$w = \frac{150}{\sin 43^\circ} = 219.9418778$

$\boxed{219.94 \text{ ft}}$

b)



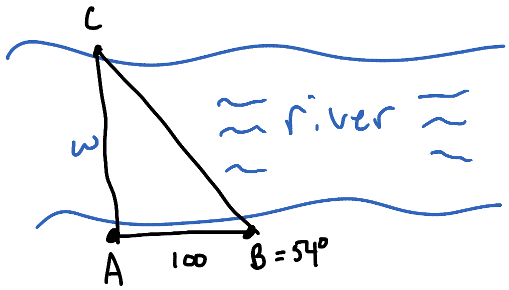
is $\tan 43^\circ = \frac{150}{g}$

$g \tan 43^\circ = 150$

$g = \frac{150}{\tan 43^\circ} = 160.8533065$

$\boxed{160.85}$ ft

Example: A biologist wants to know the width *w* of a river so that instruments for studying the pollutants in the water can be set properly. From a point A, the biologist walks downstream 100 feet and sights to point C. From this sighting, it is determined that $\theta = 54^\circ$. How wide is the river?



$$\tan 54^\circ = \frac{w}{100}$$

$$100 \tan 54^\circ = w$$

$$137.638192 = w$$

$$\boxed{137.6 \text{ ft}}$$