5.4 Sum and Difference Formulas

Sum and Difference Formulas

$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$
$$\sin(u-v)=\sin u\cos v-\cos u\sin v$$
$$\cos(u+v)=\cos u\cos v-\sin u\sin v$$
$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$
$$\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$$
$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

Examples: Find the exact value of each expression.

1. $\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

2. $\sin(135^\circ - 30^\circ) = \sin 135^\circ \cos 30^\circ - \cos 135^\circ \sin 30^\circ$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

Examples: Find the exact values of the sine, cosine, and tangent of the angle.

1. $\frac{11\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{6}$

$$\sin\left(\frac{11\pi}{12}\right) = \sin\left(\frac{3\pi}{4}\right)\cos\frac{\pi}{6} + \cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{-\sqrt{2}}{2}\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\cos\left(\frac{11\pi}{12}\right) = \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) = -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\tan\left(\frac{11\pi}{12}\right) = \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)} = \frac{-1 + \frac{1}{\sqrt{3}}}{1-(-1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{-\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{-\sqrt{3}+1}{\sqrt{3}+1}$$

2. $-\dfrac{17\pi}{12}$

first we must find two known angles to make $\frac{-17\pi}{12}$. The negative indicates it will be subtraction: $\frac{-17\pi}{12} = \frac{\pi}{4} - \frac{5\pi}{3}$

$$\sin\left(\frac{-17\pi}{12}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{3}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{-\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$\cos\left(\frac{-17\pi}{12}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{3}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{-\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$\tan\left(\frac{-17\pi}{12}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{5\pi}{3}\right)} = \frac{1-(-\sqrt{3})}{1+1(-\sqrt{3})} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

3. -105°

$-105^\circ = 45^\circ - 150^\circ$

You try it!

Examples: Write the expression as the sine, cosine, or tangent of an angle.

1. $\sin 3 \cos 1.2 - \cos 3 \sin 1.2$

Sine and cosine mixed $\Rightarrow$ use sine formula
subtract with sine $\Rightarrow$ difference formula

$\sin 3 \cos 1.2 - \cos 3 \sin 1.2 = \sin(3-1.2) = \sin(1.8)$

2. $\cos 130^\circ \cos 40^\circ - \sin 130^\circ \sin 40^\circ$

Cosines grouped + sines grouped $\Rightarrow$ cosine formula
subtract with cosine $\Rightarrow$ sum formula

$\cos 130^\circ \cos 40^\circ - \sin 130^\circ \sin 40^\circ = \cos(130^\circ + 40^\circ) = \cos 170^\circ$

3. $\dfrac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$

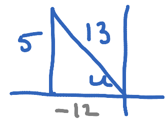
tangents $\Rightarrow$ tangent formula
sign of numerator $\Rightarrow$ difference formula

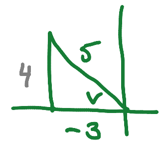
$$\frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} = \tan(45^\circ - 30^\circ) = \tan 15^\circ$$

Examples: Find the exact value of the trig function given that $\sin u = \frac{5}{13}$ and $\cos v = -\frac{3}{5}$. Both *u* and *v* are in quadrant II.

Use given info to draw triangles

find missing sides





1. $\cos(u+v)$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$= -\frac{12}{13}\left(\frac{-3}{5}\right) - \frac{5}{13}\left(\frac{4}{5}\right) = \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$$

2. $\sin(u-v)$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$= \frac{5}{13}\left(\frac{-3}{5}\right) - \left(\frac{-12}{13}\right)\left(\frac{4}{5}\right) = \frac{-15}{65} + \frac{48}{65} = \frac{33}{65}$$

3. $\sec(v-u)$

$$\sec(v-u) = \frac{1}{\cos(v-u)} = \frac{1}{\frac{56}{65}} = \frac{65}{56}$$

$$\cos(v-u) = \cos v \cos u + \sin v \sin u$$
$$= \left(-\frac{3}{5}\right)\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right) = \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$$

Examples: Prove the identity.

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$$\text{LHS} = \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x$$
$$= 1 \cdot \cos x - 0 \cdot \sin x$$
$$= \cos x \ \checkmark$$

2. $\cos\left(\frac{5\pi}{4}-x\right)=-\frac{\sqrt{2}}{2}(\cos x+\sin x)$

$LHS = \cos\left(\frac{5\pi}{4}\right)\cos x + \sin\left(\frac{5\pi}{4}\right)\sin x$

$= -\frac{\sqrt{2}}{2}\cos x + -\frac{\sqrt{2}}{2}\sin x$

$= -\frac{\sqrt{2}}{2}(\cos x + \sin x)$ ✓

Examples: Find all solutions in [0,2π).

1. $\sin(x+\pi)-\sin x-1=0$

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$\sin x\cos\pi + \cos x\sin\pi - \sin x - 1 = 0$

$\sin x(-1) + \cos x(0) - \sin x - 1 = 0$

$-2\sin x - 1 = 0$

$\to -2\sin x = 1$

$\sin x = -\frac{1}{2}$

quad 3+4

ref $\pi/6$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

2. $\tan(x+\pi)+2\sin(x+\pi)=0$

$\frac{\tan x + \tan\pi}{1-\tan x\tan\pi} + 2(\sin x\cos\pi + \cos x\sin\pi) = 0$

$\frac{\tan x + 0}{1-\tan x(0)} + 2(-\sin x) = 0$

$\tan x - 2\sin x = 0$

$\frac{\sin x}{\cos x} - 2\sin x = 0$

$\sin x\left(\frac{1}{\cos x} - 2\right) = 0$

$\sin x = 0$ when $x = 0, \pi$

$\frac{1}{\cos x} - 2 = 0$

$\frac{1}{\cos x} = 2$ so $\cos x = \frac{1}{2}$

q 1+4

ref $\frac{\pi}{3}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$