Chapter Eight: Matrices and Determinants

8.1 Matrices and Systems of Equations.

Fact: A matrix is a rectangular array of numbers. It is an ordered way to keep our coefficients and solutions organized and eliminate the need for actual variables.

Definition of a Matrix – If *m* and *n* are positive integers, an *m x n* (read *m* by *n*) matrix is a rectangular array with *m* rows and *n* columns

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

in which each entry a_{ij} of the matrix is a number. Matrices are usually denoted by capital letters.

Fact: A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the augmented matrix of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the coefficient matrix of the system.

Examples: Determine the order of the matrix.

1. $\begin{bmatrix} 5 & -3 & 8 & 7 \end{bmatrix}$

rows = 1
columns = 4
order = 1x4

2. $\begin{bmatrix} -3 & 7 & 15 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 6 & 7 \end{bmatrix}$

rows = 3
columns = 4
order 3x4

3. $\begin{bmatrix} -7 & 6 & 4 \\ 0 & -5 & 1 \end{bmatrix}$

rows = 2
columns = 3
order 2x3

Examples: Write the augmented matrix for the system of linear equations.

1. $\begin{cases} 7x+4y=22 \\ 5x-9y=15 \end{cases}$

$$\left[\begin{array}{cc|c} 7 & 4 & 22 \\ 5 & -9 & 15 \end{array}\right]$$

2. $\begin{cases} -x-8y+5z=8 \\ -7x \quad -15z=-38 \\ 3x-y+8z=20 \end{cases}$

$$\left[\begin{array}{ccc|c} -1 & -8 & 5 & 8 \\ -7 & 0 & -15 & -38 \\ 3 & -1 & 8 & 20 \end{array}\right]$$

3. $\begin{cases} 9x+2y-3z=20 \\ \quad -25y+11z=-5 \end{cases}$

$$\left[\begin{array}{ccc|c} 9 & 2 & -3 & 20 \\ 0 & -25 & 11 & -5 \end{array}\right]$$

Examples: Write the system of linear equations represented by the augmented matrix. Use variables x, y, z, and w, if applicable.

1. $\left[\begin{array}{cc|c} 7 & -5 & 0 \\ 8 & 3 & -2 \end{array}\right]$

$7x - 5y = 0$
$8x + 3y = -2$

2. $\left[\begin{array}{ccc|c} 4 & -5 & -1 & 18 \\ -11 & 0 & 6 & 25 \\ 3 & 8 & 0 & -29 \end{array}\right]$

$4x - 5y - z = 18$
$-11x + 6z = 25$
$3x + 8y = -29$

In 7.3 we discussed elementary operations for systems of equations. We will now call these row operations and use the same principles on matrices.

Elementary Row Operations –

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Row-Echelon Form and Reduced Row-Echelon Form – A matrix in ref has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row echelon form (ref) is in reduced row echelon form (rref) if every column that has a leading 1 has zeros in every position above and below except its leading 1.

Fact: When using Gaussian elimination to obtain a matrix in ref or rref, you should always operate from left to right by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

It is extremely important to have an organized approach. Left to right, top to bottom.

Examples: Write the matrix in row-echelon form.

1. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

We have a '1' in top left, let's make all zeros below it.

$-3R1$	-3	-6	3	-9
$+R2$	3	7	-5	14
NewR2	0	1	-2	5

$2R1$	2	4	-2	6
$+R3$	-2	-1	-3	8
NewR3	0	3	-5	14

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & 5 \\ 0 & 3 & -5 & 14 \end{bmatrix}$$

We have a '1' in row 2, column 2 so let's make zeros above + below

$-2R2$	0	-2	4	-10
$+R1$	1	2	-1	3
NewR1	1	0	3	-7

$-3R2$	0	-3	6	-15
$+R3$	0	3	-5	14
NewR3	0	0	1	-1

$$\begin{bmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At this point we could solve for z then back-substitute to find x and y. OR we could finish by making zeros above the '1' in r3c3.

$-3R3$	0	0	-3	3
$+R1$	1	0	3	-7
NewR1	1	0	0	-4

$2R3$	0	0	2	-2
$+R2$	0	1	-2	5
NewR2	0	1	0	3

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We can now see the solution is $(-4, 3, -1)$ with no back-substitution required.

Notes:
- Always start by making the necessary '1'.
- Use the row with the leading '1' to make your zeros.
- This method works for <u>every</u> system.
- We used rref so the answer would be displayed. You can stop earlier to find the answer if you prefer.

2. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

$$\begin{array}{r|rrrr} 3R1 & 3 & -9 & 0 & -21 \\ +R2 & 3 & 10 & 1 & 23 \\ \hline \text{New}R2 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{r|rrrr} -4R1 & -4 & 12 & 0 & 28 \\ +R3 & 4 & -10 & 2 & -24 \\ \hline \text{New}R3 & 0 & 2 & 2 & 4 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 0 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \end{bmatrix} \qquad \begin{array}{r|rrrr} 3R2 & 0 & 3 & 3 & 6 \\ +R1 & 1 & -3 & 0 & -7 \\ \hline \text{New}R1 & 1 & 0 & 3 & -1 \end{array} \qquad \begin{array}{r|rrrr} -2R2 & 0 & -2 & -2 & -4 \\ +R3 & 0 & 2 & 2 & 4 \\ \hline \text{New}R3 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{dependent system!}$$

$x + 3z = -1$

$y + z = 2$

Let $z = a$ then $x = -1 - 3a$

$y = 2 - a$

Solution: $(-1-3a, 2-a, a)$

Examples: Use matrices to solve the system of equations. Use Gaussian elimination with back-substitution or Gauss-Jordan elimination.

1. $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

$$\begin{bmatrix} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{bmatrix} \quad \tfrac{1}{2}R_1 \Rightarrow \begin{bmatrix} 1 & -1/2 & 3/2 & 12 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{bmatrix}$$

$$\begin{array}{r|rrrr} -7R_1 & -7 & \frac{7}{2} & \frac{-21}{2} & -84 \\ +R_3 & 7 & -5 & 0 & 6 \\ \hline \text{New}R_3 & 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{array} \qquad \begin{bmatrix} 1 & -1/2 & 3/2 & 12 \\ 0 & 2 & -1 & 14 \\ 0 & -3/2 & -21/2 & -78 \end{bmatrix}$$

$$\tfrac{1}{2}R_2 = \text{new}R_2 \quad \begin{bmatrix} 1 & -1/2 & 3/2 & 12 \\ 0 & 1 & -1/2 & 7 \\ 0 & -3/2 & -\frac{21}{2} & -78 \end{bmatrix} \qquad \begin{array}{r|rrrr} \frac{3}{2}R_2 & 0 & \frac{3}{2} & \frac{-3}{4} & \frac{21}{2} \\ +R_3 & 0 & -3/2 & \frac{-21}{2} & -78 \\ \hline \text{New}R_3 & 0 & 0 & \frac{-45}{4} & \frac{-135}{2} \end{array}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 12 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & \frac{-45}{4} & \frac{-135}{2} \end{bmatrix} \qquad -\tfrac{4}{45}R_3 = \text{new}R_3 \begin{bmatrix} 1 & -1/2 & 3/2 & 12 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$x - \frac{1}{2}y + \frac{3}{2}z = 12$ so $x = 8$

$y - \frac{1}{2}z = 7$ so $y = 10$

$z = 6$

$(8, 10, 6)$

2. $\begin{cases} 2x+2y-z=2 \\ x-3y+z=-28 \\ -x+y \quad =14 \end{cases}$ $\nearrow$ $\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 2 & 2 & -1 & 2 \\ -1 & 1 & 0 & 14 \end{array}\right]$

$\begin{array}{r|rrrr} -2R_1 & -2 & 6 & -2 & 56 \\ +R_2 & 2 & 2 & -1 & 2 \\ \hline \text{new}R_2 & 0 & 8 & -3 & 58 \end{array}$ $\quad$ $\begin{array}{r|rrrr} R_1 & 1 & -3 & 1 & -28 \\ +R_3 & -1 & 1 & 0 & 14 \\ \hline \text{New}R_3 & 0 & -2 & 1 & -14 \end{array}$ $\quad$ $\begin{bmatrix} 1 & -3 & 1 & -28 \\ 0 & -2 & 1 & -14 \\ 0 & 8 & -3 & 58 \end{bmatrix}$

* switched $R_2 + R_3$ why?

$-\frac{1}{2}R_2 = \text{new } R_2$ $\begin{bmatrix} 1 & -3 & 1 & -28 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 8 & -3 & 58 \end{bmatrix}$ $\quad$ $\begin{array}{r|rrrr} -8R_2 & 0 & -8 & 4 & -56 \\ +R_3 & 0 & 8 & -3 & 58 \\ \hline \text{New}R_3 & 0 & 0 & 1 & 2 \end{array}$ $\quad$ $\begin{bmatrix} 1 & -3 & 1 & -28 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

last row: $z=2$

middle row: $y-\frac{1}{2}z=7$ so $y-1=7$ and $y=8$

top row: $x-3y+z=-28$ so $x-24+2=-28$ and $x=-6$ $\quad$ $(-6\ 8, 2)$

3. $\begin{cases} 3x-2y+z=15 \\ -x+y+2z=-10 \\ x-y-4z=14 \end{cases}$ $\quad$ $\begin{bmatrix} 1 & -1 & -4 & 14 \\ -1 & 1 & 2 & -10 \\ 3 & -2 & 1 & 15 \end{bmatrix}$

what operations lead to this $\rightarrow$? $\begin{bmatrix} 1 & -1 & -4 & 14 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & 13 & -28 \end{bmatrix}$

Finish to find the solution $\quad$ $(4, -2, -2)$