12.5 Related Rates

Basic Facts: If *Q* is a quantity changing over time *t*, then the derivative *dQ/dt* is the rate at which *Q* changes over time.

Solving a Related Rates Problem -

A. The Problem

1. List the related, changing quantities.

2. Restate the problem in terms of rates of change. Rewrite the problem using math notation for the changing quantities and their derivatives.

B. The Relationship

1. Draw a diagram, if appropriate, showing the changing quantities.

2. Find an equation or equations relating the changing quantities.

3. Take the derivative with respect to time of the equation(s) relating the quantities to get the derived equation(s), which relate the rates of change to the quantities.

C. The Solution

1. Substitute into the derived equation(s) the given values of the quantities and their derivatives.

2. Solve for the derivative required.

Examples:

1. The radius of a circular puddle is growing at a rate of 5 cm/sec.

$\frac{dr}{dt} = 5$ cm/sec

a. How fast is its area growing at the instant the radius is 10 cm?

b. How fast is the area growing at the instant when it equals 36 cm²?

find $\frac{dA}{dt}$ when $r=10$

$A = \pi r^2$

$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt} = \pi(2(10))(5) = 100\pi \text{ cm}^2/\text{sec}$

when $A = 36 \text{cm}^2$

$36 = \pi r^2$ so $\frac{36}{\pi} = r^2$ and $r = \sqrt{\frac{36}{\pi}} = \frac{6}{\sqrt{\pi}}$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \frac{6}{\sqrt{\pi}}(5) = \frac{60\pi}{\sqrt{\pi}} = 60\sqrt{\pi}$

2. The average cost function for the weekly manufacture of portable CD players is given by $\bar{C}(x) = 150{,}000x^{-1} + 20 + 0.01x$ dollars per player where x is the number of CD players manufactured that week. Weekly production is currently 3000 players and is increasing at a rate of 100 players per week. What is happening to the average cost?

$x = 3000$

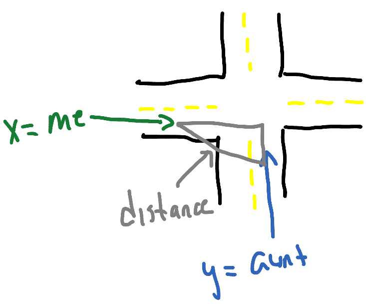
$\frac{dx}{dt} = 100$ Find $\frac{d\bar{C}}{dt}$

$$\frac{d\bar{C}}{dt} = -150{,}000\,x^{-2}\frac{dx}{dt} + 0 + 0.01\frac{dx}{dt} = \frac{-150{,}000}{x^2}\frac{dx}{dt} + 0.01\frac{dx}{dt}$$

$$\frac{d\bar{C}}{dt} = \frac{-150{,}000}{(3000)^2}(100) + 0.01(100) = -0.67$$

Average cost is decreasing by $.67 per player.

3. My aunt and I were approaching the same intersection, she from the south and I from the west. She was traveling at a steady speed of 10 miles/hour, while I was approaching the intersection at 60 miles/hour. At a certain instant in time, I was one-tenth of a mile from the intersection, while she was one-twentieth of a mile from it. How fast were we approaching each other at that instant?



$\frac{dy}{dt} = 10$

$\frac{dx}{dt} = 60$

At some time t, $x = 0.1$ $y = 0.05$

Base equation: $x^2 + y^2 = dist^2$ looking for

Derivative $\underset{0.05}{2x}\underset{10}{\frac{dx}{dt}} + \underset{0.1}{2y}\underset{60}{\frac{dy}{dt}} = 2(dist)\underset{??}{\frac{d\,dist}{dt}}$

Use base equation to find unknown distance

$(0.1)^2 + (0.05)^2 = dist^2$

$0.0125 = dist^2$

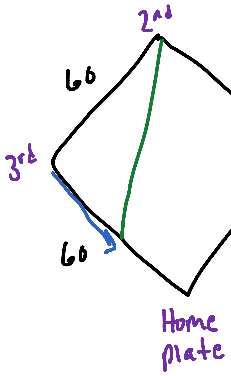
$0.1118 = dist$

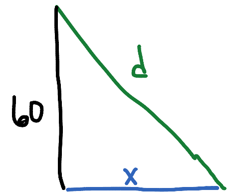
Derivative

$2(0.05)(10) + 2(0.1)(60) = 2(0.1118)\frac{dd}{dt}$

$\frac{13}{0.2236} = \frac{dd}{dt} = \boxed{58 \text{ mph}}$

4. A softball diamond is a square with sides 60 feet. A player is running from third base to home at 30 feet per second. How fast is her distance from second base increasing when she is 45 feet from third base?





$\frac{dx}{dt} = 30$ ft/sec

Find $\frac{dd}{dt}$

Base equation:

$$x^2 + 60^2 = d^2$$

Derivative $2x\frac{dx}{dt} + 0 = 2d\frac{dd}{dt}$

$$\frac{\cancel{2}x}{\cancel{2}d}\frac{dx}{dt} = \frac{dd}{dt}$$

$$\frac{45}{75}(30) = \frac{dd}{dt} = 18 \text{ ft/sec}$$

Find d:

$$45^2 + 60^2 = d^2$$

$$5625 = d^2$$

$$75 = d$$

5. The demand for personal computers in the home goes up with household income. For a given community, we can approximate the average number of computers in a home as $q = 0.3454\ln x - 3.047$ with $10,000 \le x \le 125,000$ where x is mean household income. Your community has a mean income of \$30,000, increasing at a rate of \$2,000 per year. How many computers per household are there, and how fast is the number of computers in a home increasing?

find q →

$x = 30,000$ $\quad \frac{dx}{dt} = +2000$

Find $\frac{dq}{dt}$

$$q = 0.3454\ln(30,000) - 3.047 = 0.5137 \text{ computers}$$

$$\frac{dq}{dt} = 0.3454\frac{1}{x}\frac{dx}{dt} = 0.3454\frac{1}{30\,000}(2000) = 0.0230 \text{ computers/yr}$$