Chapter 13 The Integral

13.1 The Indefinite Integral

Definition: An anti-derivative of a function *f* is a function *F* such that *F'* = *f*.

Example: An anti-derivative of $4x^3$ is x^4; an anti-derivative of $4x^3$ is x^4+2; an anti-derivative of $2x$ is x^2+11.

Fact: If the derivative of *A*(*x*) is *B*(*x*), then the anti-derivative of *B(x)* is *A*(*x*).

Definition: $\int f(x)dx$ is read "the indefinite integral of *f(x*) with respect to x" and stands for the set of anti-derivatives of *f*. Thus, $\int f(x)\,dx$ is a collection of functions; it is not a single function or a number. The function *f* that is being integrated is called the integrand, and the variable *x* is called the variable of integration.

Think about it, you have the derivative and you want to find the original function. Since the derivative of a constant is zero, we have no way of knowing what the original constant was. So we use a general C in its place and that gives us the family of functions. This is known as the constant of integration. It allows us to go from talking about 'an' anti-derivative to 'the' anti-derivatives. (Who knew an English lesson was in all this mathy stuff?)

Just like there were rules for finding derivatives, there are rules for finding anti-derivatives. These rules, by necessity, are similar to the ones we had earlier.

Power Rule

Part 1: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

Part 2: $\int x^{-1} dx = \ln|x| + C$

$\int \frac{1}{x} dx$ absolute value is absolutely important!

Exponentials

$$\int e^x dx = e^x + C$$

If *b* is any positive number other than 1, then

$$\int b^x dx = \frac{b^x}{\ln b} + C \qquad \text{i.e. } \int 2^x dx = \frac{2^x}{\ln 2} + C$$

Sums, Differences, and Constant Multiples

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

In words: the integral of a sum is the sum of the integrals (same with differences).

$\int kf(x) dx = k\int f(x) dx$ for any constant *k*.

Examples: Find the integral.

1. $\int x^7 dx$

$= \frac{x^{7+1}}{7+1} + C$

$= \frac{x^8}{8} + C$

2. $\int (-5) dx$

$= -5x + C$

what has derivative equal to -5?
$-5x$ does

3. $\int (x + x^3) dx$

$= \int x dx + \int x^3 dx$

$= \frac{x^{1+1}}{1+1} + \frac{x^{3+1}}{3+1} + C$

$= \frac{x^2}{2} + \frac{x^4}{4} + C$

4. $\int (4 - x) dx$

$= \int 4 dx - \int x dx$

$= 4x - \frac{x^{1+1}}{1+1} + C$

$= 4x - \frac{x^2}{2} + C$

5. $\int\left(\frac{1}{v^2}+\frac{2}{v}\right)dv = \int \frac{1}{v^2}dv + \int \frac{2}{v}dv = \int v^{-2}dv + 2\int \frac{1}{v}dv$

$= \frac{v^{-2+1}}{-2+1} + 2\ln|v| + C = \frac{v^{-1}}{-1} + 2\ln|v| + C$

$= -\frac{1}{v} + 2\ln|v| + C$

6. $\int\left(4x^7 - x^{-3}\right)dx$

$= 4\int x^7 dx - \int x^{-3}dx = 4\frac{x^{7+1}}{7+1} - \frac{x^{-3+1}}{-3+1} + C$

$= 4\frac{x^8}{8} - \frac{x^{-2}}{-2} + C = \frac{x^8}{2} + \frac{1}{2x^2} + C$

7. $\int\left(\frac{1}{x^{1.1}} - \frac{1}{x}\right)dx$

special case

$= \int x^{-1.1}dx - \int x^{-1}dx = \frac{x^{-1.1+1}}{-1.1+1} - \ln|x| + C$

$= \frac{x^{-0.1}}{-0.1} - \ln|x| + C = -\frac{10}{x^{0.1}} - \ln|x| + C$

Examples: Applications

1. The marginal cost of producing the xth box of thumb drives is $10+\frac{x^2}{100,000}$ and the fixed cost is \$100,000. Find the cost function C(x).

this is a derivative

$C(x) = \int\left(10 + \frac{x^2}{100,000}\right)dx = \int 10\,dx + \frac{1}{100,000}\int x^2 dx = 10x + \frac{1}{100\,000}\frac{x^3}{3} + \text{Const.}$

fixed cost is the constant

$C(x) = 10x + \frac{x^3}{300,000} + 100,000$

2. The marginal cost of producing the xth box of CDs is $10+x+\frac{1}{x^2}$. The total cost to produce 100 boxes is $10,000. Find the cost function C(x).

$C(100) = 10{,}000$

$$C(x) = \int (10 + x + x^{-2})\,dx$$

$$= 10x + \frac{x^{1+1}}{1+1} + \frac{x^{-2+1}}{-2+1} + C$$

$$= 10x + \frac{x^2}{2} + \frac{x^{-1}}{-1} + C$$

$$C(x) = 10x + \frac{x^2}{2} - \frac{1}{x} + C$$

$$10{,}000 = 10(100) + \frac{(100)^2}{2} - \frac{1}{100} + C$$

$$10\,000 = 5999.99 + C$$

$$4000.01 = C$$

$$C(x) = 10x + \frac{x^2}{2} - \frac{1}{x} + 4000.01$$

3. The velocity of a particle moving in a straight line is given by $v = 3e^t + t$.

a) Find an expression for the position after time *t*.

velocity is derivative of position so position is anti-der of velocity

$$s = \int v\,dt = \int (3e^t + t)\,dt = 3\int e^t\,dt + \int t\,dt = 3e^t + \frac{t^2}{2} + C$$

b) Given that *s = 3* at time *t = 0*, find the constant of integration C, and hence find an expression for *s* in terms of *t* without any unknown constants.

$$3 = 3e^0 + \frac{(0)^2}{2} + C$$

$$3 = 3(1) + 0 + C$$

$$3 = 3 + C$$

$$0 = C$$

so $\boxed{s = 3e^t + \frac{t^2}{2}}$