13.2 Substitution

The chain rule for derivatives gives us an extremely useful technique for finding anti-derivatives. This technique is called change of variables or substitution.

Substitution Rule - If *u* is a function of *x*, then we can use the following formula to evaluate an integral:

$$\int f dx = \int \frac{f}{\frac{du}{dx}} du$$

Rather than use the formula directly, we use the following step-by-step procedure:

1. Write *u* as a function of *x*.
2. Take the derivative *du/dx* and solve for the quantity *dx* in terms of *du*.
3. Use the expression you obtain in step 2 to substitute for *dx* in the given integral and substitute *u* for its defining expression.

Say what??? Sometimes math is easier to understand when you see it instead of just hearing the rules. General guidelines: The "inside" for integration is generally the same as for derivatives. We use the exponent for exponentials, the inside of parenthesis for powers (including radicals).

Examples: Evaluate using substitution.

1. $\int (2x+5)^{-2} dx$

Let $u = 2x+5$ this would have been "inside" for chain rule

Then $du = 2dx$

$\frac{1}{2} du = dx$

Now we substitute

$$\int (2x+5)^{-2} dx = \int u^{-2} \tfrac{1}{2} du = \frac{1}{2}\int u^{-2} du$$

$$= \frac{1}{2} \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{2u} + C$$

$$= -\frac{1}{2(2x+5)} + C$$

Always substitute twice!

once to start

once to end

2. $\int e^{-x}dx$ $= \int e^{u}(-1du) = -\int e^{u}du = -e^{u}+C$

$= -e^{-x}+C$

Let $u=-x$

$du=-1dx$

$-1du=dx$

3. $\int (x-1)^2 e^{(x-1)^3}dx$ $= \int \cancel{(x-1)^2} e^{u} \frac{1}{3\cancel{(x-1)^2}}du = \frac{1}{3}\int e^{u}du$

$= \frac{1}{3}e^{u}+C$

$= \frac{1}{3}e^{(x-1)^3}+C$

Let $u=(x-1)^3$

$du=3(x-1)^2(1)dx$

$\frac{1}{3(x-1)^2}du=dx$

4. $\int 4.1\sqrt{2x+3}dx$ $= \int 4.1\sqrt{u}\,\frac{1}{2}du = 2.05\int u^{1/2}du$

$= 2.05\frac{u^{1/2+1}}{\frac{1}{2}+1}+C$

$= 2.05\frac{u^{1.5}}{1.5}+C = \frac{2.05}{1.5}(2x+3)^{1.5}+C$

Let $u=2x+3$

$du=2dx$

$\frac{1}{2}du=dx$

5. $\int 2x\sqrt{3x^2-1}dx$ $= \int 2x\sqrt{u}\,\frac{1}{6x}du = \frac{2}{6}\int u^{1/2}du$

multiply by reciprocal

$= \frac{1}{3}\frac{u^{3/2}}{\frac{3}{2}}+C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C$

divide by fraction

$= \frac{2}{9}(3x^2-1)^{3/2}+C$

Let $u=3x^2-1$

$du=6xdx$

$\frac{1}{6x}du=dx$

Most of these problems are straight forward since the dx portion took care of any extra variables that might have been in the integral. Sometimes we have to do our substitution a little differently, we do a double substitution.

6. $\int 2x\sqrt{x+1}dx$

Let $u = x+1$, a natural "inside" choice

$du = dx$

$$\int 2x\sqrt{x+1}\,dx = \int 2x\sqrt{u}\,du$$

But what about that x? we can't have two variables

if $u = x+1$

then $x = u-1$

$$= \int 2(u-1)u^{1/2}\,du$$

Now we have only 1 variable

$$= \int (2u^{3/2} - 2u^{1/2})\,du$$

$$= 2\frac{u^{3/2+1}}{\frac{3}{2}+1} - 2\frac{u^{1/2+1}}{\frac{1}{2}+1} + C$$

$$= 2\frac{u^{5/2}}{\frac{5}{2}} - 2\frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= 2 \cdot \frac{2}{5}u^{5/2} - 2 \cdot \frac{2}{3}u^{3/2} + C$$

$$= \frac{4}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + C$$

This double substitution allowed use to use distribution to make the integral possible with *u* where it was not possible with *x*.