13.4 The Definite Integral: Algebraic Approach and the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (TFToC) - Let *f* be a continuous function defined on the interval *[a, b*] and if *F* is any anti-derivative of *f* and is defined on *[a, b],* we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Moreover, such an anti-derivative is guaranteed to exist.

In words: Every continuous function has an anti-derivative. To compute the definite integral of *f(x*) over *[a, b*], first find an anti-derivative *F(x*), then evaluate it at *x* = *b*, evaluate it at *x* = *a*, and subtract the two answers.

Examples: Evaluate the integrals.

1. $\int_{-2}^{1}(x-2)\,dx = \left(\frac{x^2}{2}-2x\right)\Big|_{-2}^{1} = \left(\frac{(1)^2}{2}-2(1)\right) - \left(\frac{(-2)^2}{2}-2(-2)\right)$

$= \left(\frac{1}{2}-2\right)-(2+4) = \frac{1}{2}-2-6 = -7.5$

the line $\Big|_{-2}^{1}$ means to evaluate

2. $\int_{0}^{1}\left(4x^3-3x^2+4x-1\right)dx$

$= \left(4\frac{x^4}{4} - 3\frac{x^3}{3} + 4\frac{x^2}{2} - 1x\right)\Big|_0^1$

$= \left(x^4 - x^3 + 2x^2 - x\right)\Big|_0^1 = \left((1)^4-(1)^3+2(1)^2-(1)\right) - \left(0^4-0^3+2(0)^2-0\right)$

$= (1-1+2-1) - 0 = \boxed{1}$

3. $\int_{2}^{3}\left(x+\frac{1}{x}\right)dx = \left(\frac{x^2}{2} + \ln|x|\right)\Big|_2^3$

$= \left(\frac{(3)^2}{2} + \ln|3|\right) - \left(\frac{(2)^2}{2} + \ln|2|\right)$

$= \frac{9}{2} + \ln|3| - 2 - \ln|2|$

$= 2.5 + \ln\left|\frac{3}{2}\right|$

Substitute twice!

4. $\int_0^1 8(-x+1)^7\,dx$ $= \int_1^0 8u^7(-1du) = -8\int_1^0 u^7 du = -8\frac{u^8}{8}\Big|_1^0 = -u^8\Big|_1^0$

1) Let $u=-x+1$

$du=-1dx$

$-1du=dx$

2) if $x=0,\ u=-0+1=1$

if $x=1,\ u=-1+1=0$

$= -(0)^8-(-(1)^8)$

$= 0+1$

$=1$

5. $\int_0^1 5xe^{x^2+2}dx$ $= \int_2^3 5xe^u \frac{1}{2x}du = \frac{5}{2}\int_2^3 e^u du$

1) Let $u=x^2+2$

$du=2xdx$

$\frac{1}{2x}du=dx$

2) $x=0,\ u=(0)^2+2=2$

$x=1,\ u=(1)^2+2=3$

$= \frac{5}{2}e^u\Big|_2^3$

$= \frac{5}{2}e^3-\frac{5}{2}e^2$

6. $\int_1^2 x(x-2)^{1/3}\,dx$ $= \int_{-1}^0 (u+2)u^{1/3}du = \int_{-1}^0 (u^{4/3}+2u^{1/3})du$

1) Let $u=x-2$

$du=dx$

$u+2=x$

2) $x=1,\ u=1-2=-1$

$x=2,\ u=2-2=0$

$= \left(\frac{u^{\frac{4}{3}+1}}{\frac{4}{3}+1}+2\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right)\Big|_{-1}^0$

$= \left(\frac{u^{7/3}}{\frac{7}{3}}+2\frac{u^{4/3}}{\frac{4}{3}}\right)\Big|_{-1}^0$

$= \left(\frac{3}{7}u^{7/3}+2\cdot\frac{3}{4}u^{4/3}\right)\Big|_{-1}^0$

$= \left(\frac{3}{7}(0)^{7/3}+\frac{3}{2}(0)^{4/3}\right)-\left(\frac{3}{7}(-1)^{7/3}+\frac{3}{2}(-1)^{4/3}\right)$

$= 0-\left(-\frac{3}{7}+\frac{3}{2}\right) = \frac{3}{7}-\frac{3}{2} = \boxed{\frac{-15}{14}}$

Whew!