Seifert surfaces for genus one hyperbolic knots in the 3-sphere

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Abstract

We prove that any collection of mutually disjoint and non-parallel genus one orientable Seifert surfaces in the exterior of a hyperbolic knot in the 3-sphere has at most 5 components and that this bound is optimal.

1 Introduction

Any knot K in the 3-sphere $\mathbb{S}^3$ bounds orientable Seifert surfaces $S' \subset \mathbb{S}^3$, and the smallest genus among such surfaces is the *genus of* K. For any minimal genus Seifert surface S' for K the once-punctured surface $S = S' \cap X_K \subset X_K$ is incompressible in the exterior $X_K = \mathbb{S}^3 \setminus \operatorname{int} N(K)$ of K, with boundary slope the *standard longitude* $J = \partial S \subset \partial X_K$ of K.

The knot K is *hyperbolic* if its complement $\mathbb{S}^3 \setminus K$ admits a complete hyperbolic structure of finite volume, or equivalently, by Thurston's work [17], if any properly embedded annulus or closed torus in its exterior X_K is compressible or parallel to ∂X_K, in which case there are at most finitely many *exceptional* slopes $r \subset \partial X_K$ for which the surgery manifold $X_K(r) = X_K \cup_\partial (\mathbb{S}^1 \times \mathbb{D}^2)$, where r bounds a meridian disk in $\mathbb{S}^1 \times \mathbb{D}^2$, is not hyperbolic.

Regarding a question of K. Motegi, of whether there is a universal bound on the number of pieces in the JSJ decomposition of the surgery manifolds $X_K(r)$ for hyperbolic knots $K \subset \mathbb{S}^3$, the family of genus one hyperbolic knots is an interesting test case. In this direction, Y. Tsutsumi [19] proved that for $r = J$ the exterior of any genus one hyperbolic knot in $\mathbb{S}^3$ contains at most 7 mutually disjoint and non-parallel genus one Seifert surfaces, providing a potential bound for the number of pieces in the JSJ decomposition of the surgery manifold $X_K(J)$, and gave an example of a genus one hyperbolic knot $K_0 \subset \mathbb{S}^3$ whose exterior contains three genus one Seifert surfaces that produce the JSJ decomposition of $X_{K_0}(J)$ consisting of three pieces, one of them hyperbolic.

In this paper we establish the optimal bound of 5 for the number of genus one Seifert surfaces in the exterior of any hyperbolic knot in $\mathbb{S}^3$.

Theorem 1. *The exterior of any genus one hyperbolic knot in* $\mathbb{S}^3$ *contains at most 5 mutually disjoint and non-parallel genus one Seifert surfaces.*

We point out that replacing the once-punctured tori in Theorem 1 with non-isotopic once-punctured Klein bottles of common boundary slope produces a similar bound (see [20, Theorem 1.1]).

Denote by $\mathbb{T}$ a collection of mutually disjoint and non-parallel once-punctured tori properly embedded in the exterior X_K of a genus one hyperbolic knot $K \subset \mathbb{S}^3$. A *complementary region* of $\mathbb{T} \subset X_K$ is the closure of a component of $X_K \setminus \mathbb{T}$ if $\mathbb{T}$ separates X_K, and otherwise the manifold X_K cut along $\mathbb{T}$. The collection $\mathbb{T} \subset X_K$ is *maximal* if it has the largest possible number of components among all such collections in X_K.

By Theorem 1, any maximal collection $\mathbb{T}$ has at most 5 components, and the next result shows that the bound of 5 is achieved by infinitely many hyperbolic knots.

Theorem 2. *There is a family of genus one hyperbolic knots*

$$K = K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6) \subset \mathbb{S}^3$$

parametrized by infinitely many choices for the integers $p_1, p_3, p_6, q_6 \geq 2$ *and* $q_1, \delta_3 \subset \{\pm 1\}$*, for each of which its exterior* X_K *contains a maximal collection of 5 mutually disjoint and non-parallel once-punctured tori, such that the JSJ decomposition of* $X_K(J)$ *consists of 5 Seifert fiber spaces over the annulus with one singular fiber and any exceptional surgery on* K *is an integral homology 3-sphere.*

All the complementary regions of $\mathbb{T} \subset X_K$ for the knots in Theorem 2 are genus two handlebodies; in fact, in Lemma 4.1 we prove that for any collection $\mathbb{T}$ at most one complementary region may not be a genus two handlebody, and if such a non-handlebody region is present then $\mathbb{T}$ has at most 4 components. Also, by Lemma 8.1 the property of any exceptional surgery on K being an integral homology 3-sphere holds for arbitrary hyperbolic knots with a 4 or 5-component collection $\mathbb{T}$ in their exterior.

The paper is organized as follows. The proofs of the main results are given in Sections 4, 7, and 8, with the remaining sections 2, 3, 5, and 6 containing supporting technical material.

The first approximation to Theorem 1 is given in Lemma 4.3, which states that any collection $\mathbb{T} \subset X_K$ has at most 6 components. Its proof relies on certain features of the complementary regions of a maximal collection $\mathbb{T}$ obtained by analyzing the properties of the disk faces of the graphs of intersection produced by $\mathbb{T}$ and a *Gabai meridional planar surface* for the knot. The complementary regions of

$\mathbb{T}$ that are handlebodies play a crucial role throughout the paper, and we model them by *pairs* (H,J) consisting of a genus two handlebody H and a separating circle $J \subset \partial H$ which is non-trivial in H and stands for the longitudinal slope of K, and in particular by *simple pairs*, which arise from boundary compressing an incompressible separating once-punctured torus in a genus two handlebody. The basic properties of pairs needed in the proof of Lemma 4.3 are presented earlier in Section 3.

In the case of a collection $\mathbb{T}$ with exactly 6 components we have that all complementary regions are genus two handlebodies; disposing of this case requires a detailed analysis of how these complementary regions fit together to form a knot exterior in $\mathbb{S}^3$, and to this end we further develop the properties of pairs in Section 6, along with some useful properties of once and twice-punctured tori in knot exteriors given in Section 5 and aimed at distinguishing satellite knots.

In Section 6.1 we show that any simple pair identifies a unique 'core knot' of its handlebody. The results of Sections 5 and 6 along with the classification of hyperbolic knots with non-integral toroidal surgeries [9] are then used to establish a mechanism in Section 7.1 by which the 'core knot' of a simple pair complementary region of $\mathbb{T}$ can be identified as a hyperbolic Eudave-Muñoz knot, whose surgery properties lead to the construction in Section 7.2 of genus two Heegaard splittings of $\mathbb{S}^3$ associated to any 6-component collection $\mathbb{T} \subset X_K$, with the knot K embedded as a separating circle in the corresponding genus two Heegaard surface. The picture obtained at this point is that of each complementary region of $\mathbb{T}$ being a simple pair, with the collection of associated core knots 'orbiting' around the knot K (see Fig. 12).

These Heegaard splittings are translated in Section 7.3 into Heegaard diagrams and further into presentations of the fundamental group of the 3-manifold corresponding to each splitting. Two nonequivalent families of Heegaard diagrams are obtained and discussed in detail in Sections 7.4 and 7.5. A theorem of T. Kaneto [15] on the structure of the relators of a group presentation of $\pi_1(\mathbb{S}^3)$ obtained from a genus two Heegaard diagram provides the final contradiction that proves Theorem 1 at the end of Section 7.5.

Section 8 is devoted to the construction of the family of genus one hyperbolic knots $K^{(1)}(p_1,q_1,p_3,\delta_3,p_6,q_6) \subset \mathbb{S}^3$ with exterior containing a 5-component collection $\mathbb{T}$ and the proof of Theorem 2. These examples are constructed by adapting some of the Heegaard splittings obtained in Section 7 so as to produce the manifold $\mathbb{S}^3$ and using a criterion from Lemma 8.1 to establish their hyperbolicity, a strategy that also allows the construction of examples of hyperbolic knots with maximal 4-component collections $\mathbb{T}$.

Interestingly, for the examples of knots where $\mathbb{T}$ has 5 components, we prove in Lemma 8.3 that the 'core knot' of at least one of the complementary regions is a hyperbolic Eudave-Muñoz knot, while conversely E. Ramírez-Losada (personal communication) has independently constructed infinite families of hyperbolic knots that bound 5 genus one Seifert surfaces starting from a tangle decomposition whose double branched cover is a hyperbolic Eudave-Muñoz knot.

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2 Preliminaries

We work in the PL category. Standard definitions, constructions and results of 3-manifold topology can be found in [10, 11], and information on JSJ decompositions of 3-manifolds in [11, 13, 14]. If A is a set or a space then $|A|$ denotes its cardinality or the number of its connected components.

Unless otherwise stated, all manifolds are assumed to be compact and orientable, and submanifolds to be properly embedded. If A is a submanifold of a manifold M then $N(A)$, $\operatorname{int}(A)$, $\operatorname{cl}(A)$, $\operatorname{fr}(A)$ denote its regular neighborhood, interior, closure, and frontier in M, respectively; the components of ∂A are denoted by $\partial_1 A, \partial_2 A, \dots, \partial_k A$. Any two submanifolds can be isotoped so as to intersect *minimally*, that is, transversely and in the smallest possible number of components.

For circles α, β in a surface S, α is non-trivial if it does not bound a disk in the surface, the isotopy class of α in the surface is called its slope (relative to the surface), $\Delta(\alpha, \beta)$ denotes their minimal geometric intersection number, and $\alpha \cdot \beta$ their integral algebraic intersection number whenever the surface S is orientable.

Let S be a surface in a 3-manifold M which is not a disk or 2-sphere. The surface S is compressible if some non-trivial circle in S bounds a disk in M, called a compression disk for S; otherwise S is incompressible. Such a surface S is boundary compressible in M if there is an arc α in S which is not boundary parallel and an arc β in ∂M with $\beta \cap S = \partial \alpha$ and not parallel in ∂M into ∂S, such that the circle $\alpha \cup \beta$ bounds a disk in M with interior disjoint from S; otherwise S is boundary incompressible. The surface S is *essential* in M if it is incompressible, boundary incompressible, and not parallel to any component of ∂M.

A 3-manifold M is irreducible if every 2-sphere in M bounds a 3-ball, and boundary irreducible if ∂M is an incompressible surface in M; M is atoroidal if any incompressible torus in M is parallel to ∂M, and toroidal otherwise. For $\Lambda \subset \partial M$ a 1-submanifold, $M(\Lambda)$ denotes the 3-manifold obtained by attaching 2-handles to M along the components of Λ and capping off any resulting 2-sphere boundary components with 3-balls. If S is a surface in M with $\partial S \neq \emptyset$, $\widehat{S}$ denotes the surface in $M(\partial S)$ obtained by capping off the circles ∂S with disjoint disks in $M(\partial S)$. We denote by $M|S$ the manifold $\mathrm{cl}\,[M \setminus N(S)] \subset M$ obtained by cutting M along S.

If $K \subset \mathbb{S}^3$ is a knot with exterior $X_K \subset \mathbb{S}^3$ then the slopes in ∂X_K correspond homologically to circles in ∂X_K of the form $p\mu + q\lambda$, where $p, q \in \mathbb{Z}$ are relatively prime integers and μ, λ are a standard meridian-longitude pair of K; we also say that $p\mu + q\lambda$ has slope $p/q \in \mathbb{Q} \sqcup \{\infty\}$, with ∞ corresponding to the slope 1/0 of μ; thus a slope $r \subset \partial X_K$ is integral iff $\Delta(r, \mu) = 1$. The knot K is simple if its exterior X_K is atoroidal, and a satellite knot otherwise; by [17] a simple knot is either a torus knot or a hyperbolic knot.

$S(n_1, \ldots, n_k)$ denotes a Seifert fiber space over the surface S with $k \geq 1$ singular fibers of indices $n_i \geq 2$. Usually S will be the 2-sphere $\mathbb{S}^2$, the disk $\mathbb{D}^2$, or the annulus $\mathbb{A}^2$. We write $S(*, \ldots, *)$ when the specific values of the n_i's are not relevant. We use L_p, $p \geq 0$, to denote a lens space with fundamental group $\mathbb{Z}/p\mathbb{Z}$, so $L_0 = \mathbb{S}^1 \times \mathbb{S}^2$ and $L_1 = \mathbb{S}^3$.

2.1 Graphs of intersection

Let M be an irreducible 3-manifold with boundary and P, Q compact surfaces (orientable or not) properly embedded in M. After isotoping P in M so as to intersect Q minimally, each component of ∂P intersects each component of ∂Q minimally in ∂M and no circle component of $P \cap Q$ is trivial in both P and Q.

We call $G_P = P \cap Q \subset P$ and $G_Q = P \cap Q \subset Q$ the *graphs of intersection* between P and Q, where we take the boundary circles of, say, P, as the *fat* vertices of G_P and the arc components of $P \cap Q$ as the edges of G_P.

If F is a face of G_P then each boundary component of F which is not a circle in $P \cap Q$ is an alternating union of edges of G_P and arcs in ∂M; F is a *k-sided face* if its boundary contains a total of k edges.

A disk face D of G_P is *trivial* if it is 1-sided. An edge of G_P is *trivial* if it is part of a trivial disk face of G_P, and *essential* otherwise. The graph G_P is *essential* if it has no trivial edges.

The faces of the graphs of G_P, G_Q can be used to find information about the complementary regions of P or Q in M; we have for instance the following well known facts.

Lemma 2.1. *1. If P is boundary incompressible then the graph G_Q is essential.*

2. *If P is incompressible then any circle component of $P \cap Q$ is non-trivial in Q.*

3. *Suppose that P is a separating surface. Let R be the closure of some component of $M \setminus P$ and D a k-sided disk face of G_Q properly embedded in R. If the graph G_P is essential then ∂D intersects $\partial P \subset \partial R$ minimally in $2k$ points; in particular D is a compression disk for ∂R in R.* □

2.2 Essential surfaces in knot exteriors

Let K be a non-trivial knot in $\mathbb{S}^3$ and P an essential surface (not necessarily orientable nor connected) in the exterior $X_K \subset \mathbb{S}^3$ of K with boundary slope $r \neq \mu$, where $\mu \subset \partial X_K$ is the meridional slope of K.

In this context, using thin position, D. Gabai proved in [7] the following result:

Lemma 2.2. *([7, Lemma 4.4]) There is a planar surface $Q \subset X_K$ with meridional boundary slope which intersects P minimally so that each arc component of $P \cap Q$ is essential in P and Q and each circle component of $P \cap Q$ is essential in Q.* □

We call the surface Q in the above lemma a *Gabai meridional planar surface for P*.

2.3 Planar graphs

A *planar graph* is a graph in a many punctured 2-sphere $Q \subset \mathbb{S}^2$.

Let G be a planar graph consisting of a set V of vertices and a set E of edges. For convenience, we also denote by V and E the cardinalities of the sets V and E, respectively, and by d the number of disk faces of G; we thus have the Euler relation $E \leq V + d - 2$.

A *bigon* is a 2-sided disk face of G. A graph without bigons is called *reduced*. We denote by $\overline{G}$ the *reduced graph of G* obtained by amalgamating each maximal collection of mutually parallel edges of G into a single edge. Thus each edge $\bar{e}$ of $\overline{G}$ corresponds to some collection of mutually parallel edges $e_1, \ldots, e_k$ of G, in which case we say that $\bar{e}$ has *size* $|\bar{e}| = k$.

Following [21], we will say that a component Γ of G is *extremal* if Γ is contained in a disk $D \subset \mathbb{S}^2$ which is disjoint from $G \setminus \Gamma$, and that a vertex v is an *interior vertex* of the extremal component Γ if v is a vertex in Γ and there is no arc in D that connects v to ∂D and whose interior is disjoint from Γ. Notice that any graph G has at least one extremal component, and that any face of G which is incident to an interior vertex of an extremal component is a disk.

Lemma 2.3. *If G is a reduced essential planar graph such that each vertex has degree at least 3 then*

1. *any extremal component of G has at least one interior vertex,*
2. *if no disk face of G is 3-sided or 5-sided then G has vertices of degree 3 and 4-sided disk faces.*

Proof. Part (1) is the content of [21, Lemma 3.2]. For part (2), let k be the smallest degree of the vertices of G and ℓ the smallest number of edges around a disk face of G. By hypothesis we have that $k \geq 3$ and $\ell = 4$ or $\ell \geq 6$, and from Euler's relation for G that $kV \leq 2E \leq 2V + 2d - 4$ and $\ell d \leq 2E \leq 2V + 2d - 4$. Therefore $(k-2)V < 2d$ and $(\ell - 2)d < 2V$, which implies that $(k-2)(\ell-2) < 4$ and hence that $k = 3$ and $\ell = 4$. □

3 Genus two handlebodies and pairs

In this section we present several properties of circles in the boundary of a genus two solid handlebody H and their relations to annuli and once-punctured tori in H, and introduce the notion of a pair (H, J).

3.1 Companion annuli and power circles in genus two handlebodies

Let M be a 3-manifold with boundary and $\gamma \subset \partial M$ a circle which is non-trivial in M. We say that a separating annulus A properly embedded in M is a *companion annulus of* γ if A is not parallel into ∂M and the circle components of ∂A cobound an annulus $A_\gamma \subset \partial M$ with core isotopic to γ in ∂M. If the region cobounded by A and A_γ in M is a solid torus V, we say that V is a *companion solid torus* of γ in M and denote the components of $M|A$ by M_A and V.

The following result gives conditions for the uniqueness in M of circles in ∂M that have companion annuli.

Lemma 3.1. *Let M be an irreducible 3-manifold with boundary and $\gamma \subset \partial M$ a separating circle that is non-trivial in M such that $\partial M = T_1 \cup_\gamma F$, where $T_1 \subset \partial M$ is a once-punctured torus. Then T_1 is incompressible in M and there is, up to isotopy, at most one circle in T_1 which has a companion annulus in M.*

Proof. Any compression of T_1 in M yields a disk in M bounded by γ, contradicting the non-triviality of the circle γ; therefore T_1 is incompressible in M.

Suppose that a, b are non-trivial circles in T_1 with incompressible companion annuli $A, B \subset H$, respectively. Isotope A and B so as intersect minimally, keeping $\partial(A \cup B) \subset T_1$, and suppose that $\partial A \cap \partial B \neq \emptyset$. Since T_1 is a once-punctured torus, each component $\partial_i A$ intersects each component $\partial_j B$ in $\Delta(\partial_i A, \partial_j B) = |\partial_i A \cdot \partial_j B|$ points; therefore the *parity rule* in [16, Lemma 2.2] applies and so any arc of $A \cap B$ has opposite parities with respect to A and B. In particular, some arc c of $A \cap B$ is positive in, say, A, and negative in B; thus c is boundary parallel in A, essential in B, and may be assumed to be outermost in A, hence to cobound with ∂A a boundary compression disk $D \subset A$ for B. Boundary compressing B along D produces a disk E properly embedded in M with $\partial E \subset T_1$ a non-trivial (separating) circle, contradicting the incompressibility of T_1. Therefore ∂A and ∂B are disjoint in T_1, so a and b are isotopic in T_1. □

We now show that each boundary component of an 'essential' annulus in a handlebody is always a non-separating circle.

Lemma 3.2. *If H is a handlebody of genus at least two and $A \subset H$ is an incompressible and non boundary parallel annulus then there is a non-trivial disk $E \subset H$ disjoint from A, with A and E both separating or both non-separating in H and each component of ∂A a non-separating circle in ∂H.*

Proof. Boundary compressing the annulus A in H yields a properly embedded non-trivial disk $E \subset H$ homologous to A which can be isotoped away from A. Thus A and E are both separating or both non-separating in H and A is isotopic to an annulus constructed by adding a band in ∂H to E along some arc $\alpha \subset \partial H$ with both endpoints on the same side of ∂E and otherwise disjoint and not parallel into ∂E, so the disk E must be non-trivial in H. As H has genus at least 2, there is a circle $\beta \subset \partial H \setminus \partial E$ which intersects α minimally in one point, which implies that each boundary component of ∂A is a non-separating circle in ∂H. $\square$

Let $\gamma, \gamma' \subset \partial H$ be mutually disjoint and non-parallel circles. We say that

- γ is a *primitive circle* in H if γ represents a primitive element in the free group $\pi_1(H)$; geometrically, this is equivalent to the presence of a disk in H which intersects γ minimally in one point;
- γ is a *power circle* in H if γ represents a non-trivial power in $\pi_1(H)$, that is, if γ represents a power $p \geq 2$ of some non-trivial element in $\pi_1(H)$ (eg, the circle L in the handlebody H_1 of Fig. 5(a));
- $\gamma, \gamma' \subset \partial H$ are *coannular* if they cobound an annulus in H, and *separated* if there is a separating non-trivial disk (a *waist* disk) in H separating γ and γ';
- $\gamma, \gamma' \subset \partial H$ are *basic circles in H* if they represent a basis of the group $\pi_1(H)$ (relative to some base point), in which case, by the 2-handle addition theorem [12, 2] applied to $\gamma' \subset \partial H \setminus \gamma$, γ and γ' must be separated circles (eg, the circles ω_1' and ω_3 in the handlebody $R_{2,3}$ of Fig. 16(a)).

The concepts above are related to Casson-Gordon's discussion in [2] of roots in the fundamental group of a compression body. The following lemmas present the results we need here in the context of genus two handlebodies and through the properties of companion annuli, which will become increasingly relevant in the sequel.

Lemma 3.3. *Let H be a genus two handlebody and $\gamma \subset \partial H$ a circle which is non-trivial in H. Then,*

1. *the surface $\partial H \setminus \gamma$ compresses in H iff γ is a primitive or a power circle in H, in which case the following conditions hold:*

   (a) *$\partial H \setminus \gamma$ compresses along a waist disk $D_w \subset H$ which cuts H into two solid tori $V, V' \subset H$ with $\gamma \subset \partial V$,*

   (b) *$\partial H \setminus \gamma$ compresses along a non-separating disk in H, which is unique up to isotopy;*

2. *γ has a companion annulus in H iff γ is a power circle in H, in which case γ represents a non-trivial power of some primitive element of $\pi_1(H)$; more precisely,*

   (a) *the companion annulus A of γ is unique up to isotopy and cobounds with ∂H a companion solid torus of γ, of whose core γ represents a non-trivial power in $\pi_1(H)$,*

   (b) *$H|A$ consists of a genus two handlebody H_A and a solid torus, and the core of A is a primitive circle in H_A.*

Proof. That $\partial H \setminus \gamma$ compresses in H iff γ is a primitive or a power circle in H, and that γ has a companion annulus in H iff γ is a power circle in H, follow from [20, Lemma 5.2].

Suppose that $D \subset H$ is a compression disk for $\partial H \setminus \gamma$. If $\partial D \subset \partial H \setminus \gamma$ is a non-separating circle then there is a circle $\alpha \subset \partial H \setminus \gamma$ which intersects ∂D transversely in one point, hence D is non-separating and the waist disk $D_w = \mathrm{fr}\, N(D \cup \alpha) \subset H$ is a compression disk for $\partial H \setminus \gamma$ which cuts H into two solid tori $V, V' \subset H$ with, say, $\gamma \subset V$, so (1)(a) and the first part of (2) hold.

If $\partial D \subset \partial H \setminus \gamma$ is a separating circle then we can take $D_w = D$ as the a waist disk for H in the above argument, so that $H = V \cup_{D_w} V'$ with $\gamma \subset \partial V$, whence $\partial H \setminus \gamma$ compresses along some meridian disk D' of the solid torus V', which is non-separating in H. It is not hard to see that D' is unique in H up to isotopy so (1)(b) holds.

In (2), any companion annulus A of γ can be isotoped away from D' and into the solid torus $H|D'$, so the uniqueness of A and the fact that it cobounds with ∂H a companion solid torus of γ follow from the uniqueness of D'. Since $H = V \cup_{D_w} V'$ and $\gamma \subset \partial V$, A may also be isotoped in H away from D_w so that $A \subset V$ runs $p \geq 2$ times around V, whence γ represents the pth power of the core circle of V, which is primitive in $\pi_1(H)$. We also have the decomposition $V|A = V_1 \cup_A V_2$ for some solid tori $V_1, V_2 \subset V$ with $\gamma \subset V_1$ and $A \subset \partial V_2$ running once around V_2; as $H_A = V_2 \cup_{D_w} V'$, it follows that H_A is a genus two handlebody and that the core of $A \subset \partial V_2$ is isotopic to the core of V_2, which is primitive in H_A. Therefore (2) holds. □

In light of Lemma 3.3(2), we will say that

- a circle $\gamma \subset \partial H$ is a *power p circle* for some integer $p \geq 2$ if γ is a power circle in H that represents the power p of some primitive element of $\pi_1(H)$, or, equivalently, if γ runs p times around its companion solid torus in H;
- we extend this notation so that a circle $\gamma \subset \partial H$ is a power $p = 1$ circle iff γ is a primitive circle in H.

Regarding separated or coannular circles we have the following result.

Lemma 3.4. *Let $\gamma, \gamma' \subset \partial H$ be disjoint and non-parallel circles in ∂H which are non-trivial in a genus two handlebody H, and let $S = \partial H \setminus (\gamma \cup \gamma') \subset \partial H$. Then S has at most one compression disk in H up to isotopy, and the following conditions hold:*

1. *the surface S compresses in H along a separating disk iff γ and γ' are separated in H, in which case each circle γ and γ' is a primitive or power circle in H,*
2. *the surface S compresses in H along a non-separating disk iff γ and γ' are coannular circles in H, in which case γ and γ' are both primitive or both power circles in H.*
3. *if γ is a primitive or power circle and γ' is a power circle then S compresses in H along a separating disk,*
4. *if γ and γ' are coannular in H and D is the non-separating compression disk for S then*
    - *(a) up to isotopy, the annulus bounded by $\gamma \sqcup \gamma'$ is unique in H when γ is a primitive circle, and there are exactly two such annuli when γ is a power circle,*
    - *(b) if γ is a power circle with companion annulus $B \subset H$ then B can be isotoped into $H|D$, in which case γ' and the core circle γ_B of B are coannular and primitive in the genus two handlebody $H_B \subset H|B$.*

Proof. If the circles γ and γ' are separated by a waist disk $D \subset H$ then D is a compression disk for S in H. If γ and γ' cobound an annulus $A \subset H$ then by Lemma 3.2 each circle γ and γ' is non-separating in ∂H, so A is non-separating and there is a non-separating disk $E \subset H$ disjoint from A, which is then a compression disk for S. In either case the surfaces $\partial H \setminus \gamma$ and $\partial H \setminus \gamma'$ compress in H and so by

Lemma 3.3 each circle γ and γ' is a primitive or a power circle in H, and it is not hard to see that D and E are unique up to isotopy.

Since the circles γ and γ' are not parallel in ∂H, if S compresses in H along a separating disk $D \subset H$ then γ and γ' are separated by D in H; thus (2) holds.

If γ is a primitive or power circle and γ' is a power circle in H then the surface $F = \partial H \setminus \gamma$ compresses in H by Lemma 3.3(1) and contains γ'; since by Lemma 3.3(2), if B' and V' are the companion annulus and solid torus of γ' then the manifold $H(\gamma') = H_B(\gamma') \cup_{\widehat{B'}} V'(\gamma')$ is a connected sum of a solid torus and a lens space, it follows by the 2-handle addition theorem that the surface $S = F \setminus \gamma'$ compresses in H. Thus (3) holds.

Suppose now that there is a non-separating compression disk $D \subset H$ for S. Then $H|D$ is a solid torus with $\gamma \sqcup \gamma' \subset \partial(H|D)$, so the closures of the components of $\partial(H|D) \setminus (\gamma \sqcup \gamma')$ are two annuli A, A' and so γ and γ' are coannular in $H|D$, hence in H. Thus (1) holds.

Let $\mathscr{A} \subset H$ be any properly embedded annulus with boundary $\gamma \sqcup \gamma'$. By Lemma 3.2, the annulus $\mathscr{A}$ is incompressible and non-separating in H, so S compresses in H along a unique non-separating disk $D \subset H$ disjoint from $\mathscr{A}$; therefore $\mathscr{A}$ lies in the solid torus $H|D$ and hence it is parallel to A or A' in $H|D$.

Let $p \geq 1$ be the number of times that γ runs around $H|D$, so that γ is a power p circle in H. If $p = 1$ then A and A' are parallel in $H|D$ and so, up to isotopy, $\mathscr{A}$ is the unique annulus in H cobounded by γ and γ'. If $p \geq 2$ then A and A' are not parallel in $H|D$ and so there are two possible such annuli $\mathscr{A}$.

Now, if $p \geq 2$ and $B \subset H$ is the companion annulus of γ then B can be isotoped so as to intersect D minimally and hence, by a standard innermost-outermost intersection argument, to be disjoint from D. Since the circles γ and γ' are not parallel in ∂H, necessarily the core circle γ_B of B and γ' are not parallel in the genus two handlebody $H_B \subset H|B$. Therefore, by Lemma 3.3(2), γ_B is a primitive circle in H_B, and by part (2) the circles γ' and γ_B are coannular in H_B. Thus (4) holds. □

The following result gives conditions for the manifold obtained by attaching one or two solid tori to a genus two handlebody to be a handlebody.

Lemma 3.5. *Let H be a genus two handlebody and $\gamma, \gamma' \subset \partial H$ a pair of disjoint circles.*

1. *If $M = H \cup_\gamma V$ is a manifold obtained by gluing a solid torus V to H along an annular neighborhood $A = \partial H \cap \partial V$ of γ, such that A runs at least twice around V, then M is a genus two handlebody iff γ is a primitive circle in H.*

2. *If $M = V \cup_\gamma H \cup_{\gamma'} V'$ is a manifold obtained by gluing solid tori V, V' to H along disjoint annular neighborhoods $A = \partial H \cap \partial V$ and $A' = \partial H \cap \partial V'$ of*

γ and γ', respectively, where each annulus A, A' runs at least twice around V, V', respectively, then M is a genus two handlebody iff γ and γ' are basic circles in H.

Proof. For part (1), if M is a genus two handlebody then by Lemma 3.3 the annulus $A \subset M$ is a companion annulus of some power circle in ∂M, so by Lemma 3.3(2)(b) the circle γ is primitive in H. Conversely, if $\gamma \subset \partial H$ is primitive in H then by Lemma 3.3 there is a waist disk of H disjoint from γ which cuts H into solid tori W, W' with $\gamma \subset W$ a circle that runs once around W; therefore $V \cup_A W$ is a solid torus, so $M = V \cup_A (W \cup_{D_w} W') = (V \cup_A W) \cup_{D_w} W'$ is a genus two handlebody.

For part (2), if M is a genus two handlebody then A and A' are companion annuli of some disjoint power circles α and α' in ∂M, respectively; since A and A' are disjoint, by [20, Lemma 5.1] the circles α and α' are not mutually parallel in ∂M. Therefore, by Lemma 3.4(1),(3) the circles α and α' are separated in M by some waist disk $D \subset M$, which can be isotoped in M to be disjoint from $A \sqcup A'$ to become a separating disk for $\gamma, \gamma' \subset \partial H$. The argument for part (1) now shows that γ and γ' are primitive and hence basic circles in H. The converse follows in a similar way. □

3.2 Pairs

A *pair* (H, J) consists of a genus two handlebody H and a separating circle $J \subset \partial H$ which is non-trivial in H. If (H, J) is a pair then the closures T_1, T_2 of the components of $\partial H \setminus J$ are once-punctured tori with $\partial T_1 = J = \partial T_2$ and $\partial H = T_1 \cup_J T_2$.

A pair (H, J) is

- *trivial* if it is homeomorphic to the pair $(T_1 \times I, \partial T_1 \times \{0\})$ with T_1 corresponding to $T_1 \times \{0\}$;
- *minimal* if any once-punctured torus T in H with $\partial T = J$ is parallel to T_1 or T_2; in particular any trivial pair is minimal;
- if $\omega_i \subset T_i$ is a power circle in H with companion annulus $A_i \subset H$, where the circles ∂A_i cobound an annulus $A_i' \subset T_i$, then isotoping $(T_i \setminus A_i') \cup A_i$ slightly off T_i produces a once-punctured torus T_i' properly embedded in H with $\partial T_i' = J$, and we say that T_i' is the once-punctured torus in H *induced* by the power circle ω_i.

The next result establishes the uniqueness of power circles in a couple of related situations.

Lemma 3.6. *Let H be a genus two handlebody and $\gamma \subset \partial H$ a non-trivial circle in H.*

1. *If γ separates ∂H into once-punctured tori T_1, T_2, then each T_i is incompressible in H and contains, up to isotopy, at most one power circle.*

2. *If the circle γ is non-separating in ∂H and neither a primitive nor power circle in H then any two circles in $\partial H \setminus \gamma$ which are power circles in H are isotopic in the torus $\partial H(\gamma)$.*

Proof. Part (1) follows from Lemma 3.1. For part (2), by Lemma 3.3(1) the surface $\partial H \setminus \gamma$ is incompressible in H, hence by the 2-handle addition theorem the manifold $H(\gamma)$ is irreducible with incompressible torus boundary. So if $a, b \subset \partial H \setminus \gamma$ are any power circles in H with corresponding companion annuli $A, B \subset H$ then the annuli A and B are essential in $H(\gamma)$ and so, by an argument similar to the one used in the proof of Lemma 3.1, their minimal intersection $A \cap B$ must be empty, whence a and b are isotopic in $\partial H(\gamma)$. □

It follows from Lemma 3.6(1) that the once-punctured torus induced by a power circle in $T_i \subset \partial H$ is unique up to isotopy. We will say that

- a pair (H, J) is *simple* if, for some $\{i, j\} = \{1, 2\}$, T_j is parallel in H to the once-punctured torus induced by some power circle in T_i.

We will see below that the pair (H, J) in Fig. 9(a) is simple.

The next result establishes several basic facts about pairs.

Lemma 3.7. *Let (H, J) be a pair with $\partial H = T_1 \cup_J T_2$ and $T \subset H$ any once-punctured torus with $\partial T = J$. Then*

1. *$H(J)$ is an irreducible manifold with incompressible boundary $\widehat{T}_1 \sqcup \widehat{T}_2$,*

2. *T is incompressible and separates H into two components whose closure are genus two handlebodies H_1, H_2 with $\partial H_i = T \cup_J T_i$.*

3. *T boundary compresses in H towards some T_i, in which case the pair (H_i, J) is either trivial or simple,*

4. *the pair (H, J) is trivial iff $H(J) \approx \widehat{T}_1 \times I$.*

Proof. By Lemma 3.6, T_1 and T_2 are incompressible in H and hence (1) holds by the 2-handle addition theorem. Similarly, T is incompressible in H. Since H can be embedded in $\mathbb{S}^3$, the closed surface $T \cup T_1$ is orientable and separates $\mathbb{S}^3$, hence T

separates H into two components whose closures $H_1, H_2 \subset H$ satisfy $H = H_1 \cup_T H_2$. That H_1 and H_2 are handlebodies follows now as in [19, Lemma 2.3].

Suppose now for definiteness that T boundary compresses in H towards T_1. Then T boundary compresses into an annulus $A \subset H$ with ∂A non-separating circles in T_1 that cobound an annulus $A_1 \subset T_1$. The once-punctured torus T can be recovered by adding a band to the annulus A along an arc in T_1 with one endpoint in $\partial_1 A$ and the other in $\partial_2 A$, that is, T is parallel in H to the once-punctured torus $(T_1 \setminus A_1) \cup A$. If A is parallel to A_1 then T is parallel to T_1, so $H_1 \approx T_1 \times I$ and hence the pair (H_1, J) is trivial. If A is not parallel to A_1 then A is a companion annulus of the core circle ω_1 of A_1, in which case, by Lemma 3.3(2), the circle ω_1 is a $p \geq 2$ power circle in H, which implies that T is parallel in H_1 to the once-punctured torus in H_1 induced by the power circle $\omega_1 \subset T_1$. Thus (3) holds.

For part (4), if $H(J)$ is homeomorphic to $\widehat{T}_1 \times I$ then J is the boundary of the cocore disk for some tunnel arc t of $\widehat{T}_1 \times I$. As H is a handlebody, by [6, Lemma 1.1] the arc t is isotopic in $\widehat{T}_1 \times I$ to a vertical arc $\{x\} \times I$ and so (H, J) is homeomorphic to the trivial pair $(T_1 \times I, \partial T_1)$. The converse follows by definition of trivial pair. □

We now construct a special family of pairs described in [19, §4]. Let F be a once-punctured torus and $\alpha_1, \alpha_2 \subset F$ properly embedded circles that intersect transversely in one point. The manifold $F \times I$ is a genus 2 handlebody with boundary $(F \times \{0\}) \cup ((\partial F) \times I) \cup (F \times \{1\})$, and the circles $\gamma_1 = \alpha_1 \times \{0\} \subset F \times \{0\}$ and $\gamma_2 = \alpha_2 \times \{1\} \subset F \times \{1\}$ form a basis of the rank two free group $\pi_1(F \times I)$. We denote by J the separating circle $(\partial F) \times \{1/2\} \subset \partial(F \times I)$. Fig. 1 shows the 4-tuple $(F \times I, J, \gamma_1, \gamma_2)$ up to homeomorphism.

Let H be the manifold obtained by gluing solid tori V_1, V_2 to $F \times I$ along annular regular neighborhoods of the circles γ_1, γ_2, respectively, so that γ_i is the fiber of a fibration of type (a_i, p_i) in V_i for some $p_i \geq 1$ (whence γ_i runs p_i times around V_i). By Lemma 3.5(2) H is a genus two handlebody.

We will call a pair (H, J) constructed as above a pair of *type* $(a_1, p_1; a_2, p_2)$, or in short of *type* (p_1, p_2); clearly, any pair of type (p_1, p_2) is also of type (p_2, p_1).

Remark 3.8. *1. A pair is trivial iff it is of type* $(1, 1)$.

2. *A pair is simple iff it is of type* $(p, 1)$ *or* $(1, p)$ *for some* $p \geq 2$ *(see Fig. 9(a)).* For if (H, J) is a $(p, 1)$ pair, with $H = (F \times I) \cup V_1$ and $J = (\partial F) \times \{1/2\}$ as above, then the core $\omega_1 \subset T_1$ of the annulus $\partial V_1 \setminus (F \times \{0\})$ is a power $p \geq 2$ circle in H with companion annulus $\partial V_1 \cap (F \times \{0\})$; thus the once-punctured torus T_1' induced by $\omega_1 \subset T_1$ in H can be identified with $F \times \{0\}$, which is parallel to $T_2 = F \times \{1\}$ in H, whence the pair (H, J) is simple. Conversely, if (H, J) is simple then we may assume that there is a circle

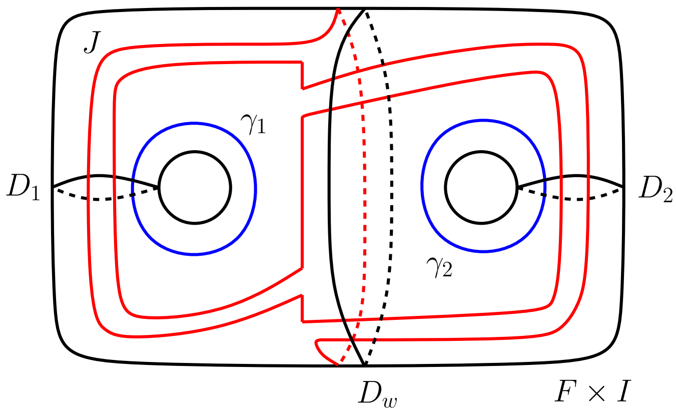

Figure 1: The genus two handlebody $F \times I$.

$\omega_1 \subset T_1$ which is a power $p \geq 2$ circle in H, with companion annulus $A \subset H$ and companion solid torus $V \subset H$, such that T_2 is parallel in H to the once-punctured torus $T_1' \subset H$ induced by ω_1. Thus the region in H cobounded by T_1' and T_2 is homeomorphic to $T_2 \times [0,1]$, with T_2 corresponding to $T_2 \times \{0\}$, T_1' to $T_2 \times \{1\}$, and J to the circle $(\partial T_2) \times \{0\}$; as H is homeomorphic to the handlebody obtained by adding the companion solid torus V of ω_1 to the core of the annulus $A \subset T_2 \times \{1\}$, by definition (H,J) is a $(p,1)$ pair.

3. *A pair of type (p_1, p_2) with $p_1, p_2 \geq 2$ will be called a double pair.*

The following result summarizes the content of Lemmas 4.2, 4.3 and 4.4 of [19].

Lemma 3.9 ([19])**.** *For any pair (H,J),*

1. *if (H,J) is simple then it is minimal,*
2. *H contains at most two once-punctured tori with boundary slope J which are mutually disjoint and non-parallel, and not parallel into ∂H.* □

In light of Lemma 3.9, we will say that

- a pair (H,J) is *maximal* if H contains two disjoint, mutually non-parallel once-punctured tori $T_1', T_2' \subset H$ with boundary slope J which are not parallel to T_1 or T_2.

In such case, by Lemma 3.7 $T_1' \cup T_2'$ cuts H into handlebodies H_0, H_1, H_2 with $\partial H_0 = T_1' \cup T_2'$ and $H = H_1 \cup_{T_1'} H_0 \cup_{T_2'} H_2$. The following result is an immediate consequence of Lemmas 3.7(3) and 3.9(1).

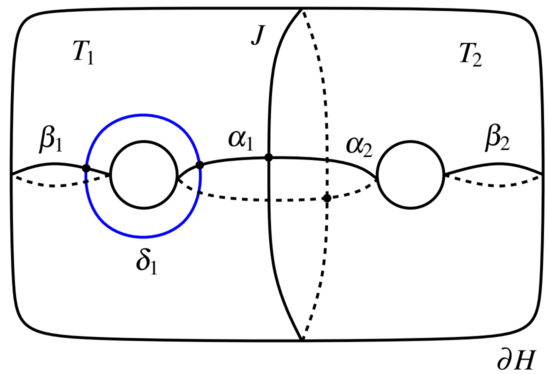

Figure 2: The circle $\delta_1 \subset T_1$.

Corollary 3.10. *If (H,J) is a maximal pair with $H = H_1 \cup_{T_1'} H_0 \cup_{T_2'} H_2$ then the pairs (H_1,J) and (H_2,J) are simple.* □

The construction of maximal pairs will be discussed in more detail in Remark 7.7. The last result of this section provides a useful classification of trivial or simple pairs.

Lemma 3.11. *A pair (H,J) is of type $(1,p)$ for some $p \geq 1$ iff there is a disk in H which intersects J minimally in 2 points.*

Proof. Suppose that (H,J) is a $(1,p)$ pair obtained from the pair $(F \times I, J)$ in Fig. 1 by gluing a solid torus V_2 along the circle $\gamma_2 \subset \partial(F \times I)$, so that γ_2 runs p times around V_2. Then the disk $D_1 \subset F \times I$ shown in Fig. 1 is properly embedded in H and intersects $J \subset \partial H$ minimally in 2 points.

Conversely, suppose $D \subset H$ is a non-trivial disk which intersects $J \subset \partial H$ minimally in 2 points, and let $\partial H = T_1 \cup_J T_2$. Then, for each $i = 1,2$, $\alpha_i = T_i \cap \partial D$ is a non-trivial, hence non-separating arc in T_i, and so D is a non-separating disk in H.

Let β_i be the core circle of the annulus obtained by cutting T_i along the arc $\alpha_i \subset T_i$. Then β_1 and β_2 are disjoint from the circle $\partial D = \alpha_1 \cup \alpha_2$, hence from D, so by Lemma 3.4 the circles β_1 and β_2 are coannular power $p \geq 1$ circles in H. We also let $\delta_1 \subset T_1$ be any circle that intersects the arc $\alpha_1 \subset T_1$ and the circle $\beta_1 \subset T_1$ each transversely in one point, so that δ_1 is primitive in H. The situation is represented in Fig. 2.

If $p \geq 2$ then by Lemma 3.4(4) the power circle β_2 has a companion annulus $B \subset H$ and companion solid torus $V_B \subset H$ disjoint from D, with core circle $\beta_2' \subset B$ such that β_2' and β_1 are coannular and primitive circles in the genus two handlebody $H_B \subset H|B$.

If $p \geq 2$ we let $H' = H_B$, and if $p = 1$ we set $H' = H$. Thus H' is a genus two handlebody with $J, \beta_1, \beta_2', \delta_1 \subset \partial H'$ and $D \subset H'$, where D intersects J minimally in two points and is disjoint from the coannular primitive circles $\beta_1, \beta_2' \subset \partial H'$.

Now, the disk $D' = \mathrm{fr}\, N(\delta_1 \cup D) \subset H'$ is a waist disk of H' that separates the primitive circles $\delta_1, \beta_2' \subset \partial H'$ and intersects $J \subset \partial H'$ minimally in 4 points. Therefore the 4-tuple $(H', J, \delta_1, \beta_2')$ is homeomorphic to the 4-tuple $(F \times I, J, \gamma_1, \gamma_2)$ of Fig. 1; since $H = H'$ for $p = 1$ and $H = H' \cup_B V_B$ for $p \geq 2$, it follows that the pair (H, J) is of type $(1, p)$. $\square$

4 Genus one hyperbolic knots in $\mathbb{S}^3$

In this section we assume that $K \subset S^3$ is a genus one hyperbolic knot and $\mathbb{T} = T_1 \sqcup \cdots \sqcup T_N$ a collection of $N = |\mathbb{T}| \geq 1$ mutually disjoint and non-parallel once-punctured tori properly embedded in X_K with boundary slope the longitude J of K, where the T_i's are labeled consecutively around $\partial N(K)$ following some fixed orientation of the meridian slope $\mu \subset \partial N(K)$, as in Fig. 3.

4.1 Complementary regions of $\mathbb{T} \subset X_K$

For any $1 \leq i, j \leq N$ with $i \neq j$ denote by $R_{i,j} \subset X_K$ the region cobounded by T_i and T_j that contains the oriented arc of μ with $\mu \cap \partial T_i$ as initial point and $\mu \cap \partial T_j$ as terminal point (see Fig. 3), so that $R_{i,j} \cap R_{j,i} = T_i \sqcup T_j$ and $X_K = R_{i,j} \cup R_{j,i}$. For $i = j$ we let $R_{i,i} = \mathrm{cl}\,[X_K \setminus N(T_i)]$ be the manifold obtained by cutting X_K along T_i.

Since the surface $\mathbb{T}$ is essential in X_K, by Lemma 2.2 there is a Gabai meridional planar surface Q for $\mathbb{T}$ which intersects $\mathbb{T}$ minimally in essential graphs $G_Q = Q \cap \mathbb{T} \subset Q$ and $G_{\mathbb{T}} = Q \cap \mathbb{T} \subset \mathbb{T}$ such that each circle component of $Q \cap \mathbb{T}$ is essential in Q. We denote the subgraph $Q \cap (T_{i_1} \sqcup \cdots \sqcup T_{i_k}) \subset Q$ of G_Q by $G_Q^{i_1,\ldots,i_k}$.

The next result establishes connections between properties of the graph G_Q and the regions $R_{i,j}$.

Lemma 4.1. *Each boundary cycle of any face of G_Q has an even number of edges, and for any i, j either $R_{i,j}$ is a genus two handlebody or an atoroidal irreducible and boundary irreducible manifold. In particular, the following regions are genus two handlebodies:*

1. *at least one of the regions $R_{i,j}$ or $R_{j,i}$ for any $i \neq j$,*
2. *any region $R_{i,j}$ that contains a disk face of $G_Q^{i,j}$ (with $i = j$ allowed),*
3. *any region $R_{i,i+1}$ if G_Q is connected or each vertex of $\overline{G}_Q$ has degree at least 3, and if some region $R_{i,i+1}$ is not a handlebody then $|\mathbb{T}| \leq 4$ and any other region $R_{j,j+1}$ is a handlebody.*

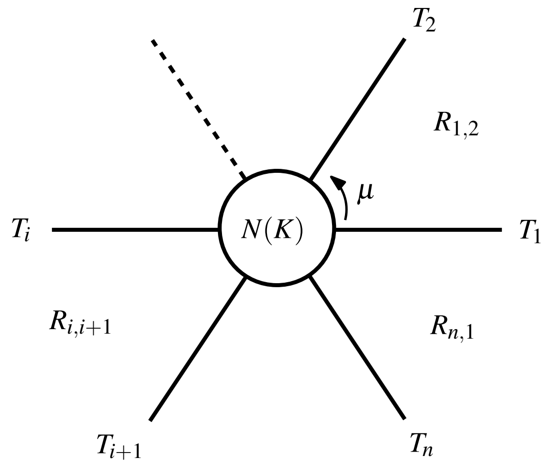

Figure 3: The once-punctured tori $T_i \subset X_K$.

Proof. That each boundary cycle of any face of G_Q is even sided follows from the fact that each component T_i of $\mathbb{T}$ has one boundary component. As K is a hyperbolic knot, its exterior $X_K \subset \mathbb{S}^3$ is irreducible and atoroidal, and since T_i and T_j are incompressible in X_K each region $R_{i,j}$ is irreducible and atoroidal too.

Since the boundary slope J of T_i and T_j is a longitude of K, in $\mathbb{S}^3$ the surfaces $\partial R_{i,j}$ and $\partial R_{j,i}$ for $i \neq j$ or $\partial R_{i,i}$ and $\partial N(T_i)$ for $i = j$ are mutually parallel and hence compressible. If, say, $\partial R_{i,j}$ compresses in $R_{i,j}$ then the maximal compression body W of $\partial R_{i,j}$ in $R_{i,j}$ with $\partial_+ W = \partial R_{i,j}$ (see [1]) is non-trivial and so either $\partial_- W = \emptyset$ or $\partial_- W$ is a collection of incompressible closed tori in $R_{i,j}$. As T_i and T_j are incompressible surfaces in X_K, any torus component of $\partial_- W$ must be incompressible in X_K, contradicting the hyperbolicity of K; therefore $\partial_- W = \emptyset$, so $W = R_{i,j}$ is a genus two handlebody, and so (1) holds.

Part (2) follows now from Lemma 2.1 and the argument above. If G_Q is connected then all its faces are disks, while if each vertex of the reduced graph $\overline{G}_Q$ has degree at least 3 then by Lemma 2.3 any extremal component of $\overline{G}_Q$ has an interior vertex v_0, whence all faces of G_Q incident to v_0 must be disks; in either case we have that necessarily each region $R_{i,i+1}$ contains a disk face of G_Q, so the first part of (3) follows from (2), and the second part is now a consequence of (1) and Lemma 3.9(2). □

Lemma 4.2. *If for some $i \neq j$ the region $R_{i,j}$ contains a bigon disk face of $G_Q^{i,j}$ then $R_{i,j}$ is a handlebody and the pair $(R_{i,j}, J)$ is simple. In particular $|\overline{e}| \leq 2$ for each edge $\overline{e}$ of $\overline{G}_Q$.*

Proof. Suppose that $D \subset R_{i,j}$ is a bigon face of $G_Q^{i,j}$; in particular, D may be the bigon disk face $\overline{D}$ in G_Q cobounded by the outermost edges $e_i \subset T_i$ and $e_j \subset T_j$ of some edge $\overline{e} = \{e_i, e_{i+1}, \ldots, e_j\}$ of $\overline{G}_Q$ with $|\overline{e}| \geq 2$. By Lemma 4.1(2) the region $R_{i,j}$ is a handlebody and so $(R_{i,j}, J)$ is a non-trivial pair, while by Lemma 2.1 the disk $D \subset R_{i,j}$ is non-trivial and intersects $\partial T_i \sqcup \partial T_j$ minimally in 4 points, hence J minimally in two points. Therefore by Lemma 3.11 the pair $(R_{i,j}, J)$ is simple and hence minimal by Lemma 3.9(1), which in the case of $D = \overline{D}$ implies that $j = i+1$ and hence that $|\overline{e}| = 2$. □

We now establish a first approximation to Theorem 1.

Lemma 4.3. *If $K \subset \mathbb{S}^3$ is a genus one hyperbolic knot and $\mathbb{T} = T_1 \sqcup \cdots \sqcup T_N \subset X_K$ is a collection of $N \geq 1$ mutually disjoint and non-parallel once-punctured tori then $N \leq 6$, and if $N \geq 5$ then each complementary region $R_{i,i+1}$ is a handlebody.*

Proof. By Lemma 2.2, there is a Gabai meridional planar surface $Q \subset X_K$ for $\mathbb{T}$ which intersects $\mathbb{T}$ minimally in essential graphs $G_Q \subset Q$ and $G_{\mathbb{T}} \subset \mathbb{T}$, such that each vertex of the graph G_Q has degree N and, by Lemma 4.1, each disk face of G_Q, and hence of its reduced graph $\overline{G}_Q$, has an even number of edges around its boundary. Therefore, by Lemma 4.2 each vertex of $\overline{G}_Q$ has degree at least $N/2$. If $N \geq 5$ then each vertex of $\overline{G}_Q$ has degree at least 3 and so, by Lemma 2.3(2), $\overline{G}_Q$ has a vertex of degree $3 \geq N/2$, so $N \leq 6$, and each region $R_{i,i+1}$ is a handlebody by Lemma 4.1(3). □

In the next couple of sections we digress to present the supporting results needed for the analysis in Section 7 of the case $|\mathbb{T}| = 6$ and the construction in Section 8 of examples of hyperbolic knots for the cases $|\mathbb{T}| = 4, 5$.

5 Toroidal surfaces in knot exteriors

The results in this section analyze the interaction between once or twice-punctured tori in a satellite knot exterior in $\mathbb{S}^3$ and the companion annuli of circles in such surfaces, and will be used in §7.1 to establish the connection between hyperbolic knots in $\mathbb{S}^3$ with 6-component collections $\mathbb{T}$ and the family of hyperbolic Eudave-Muñoz knots.

5.1 Once-punctured tori in X_K

We extend the definition of companion annulus given in §3.1 to include the case of circles in non-separating orientable surfaces.

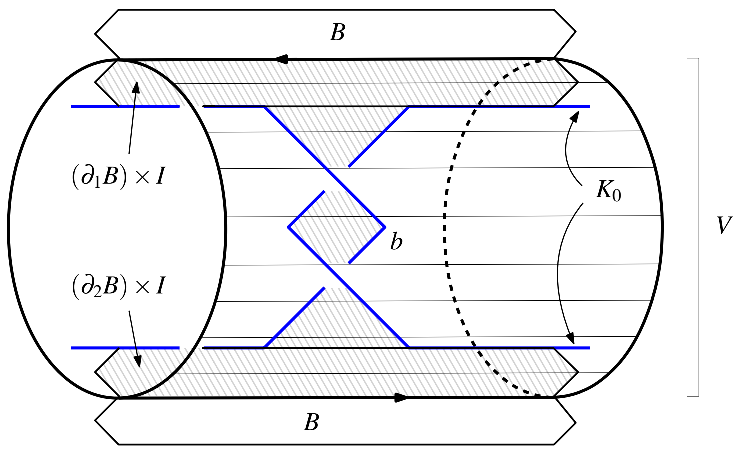

Figure 4: The knot K_0 as a boundary component of the pair of pants $P = (\partial B) \times I \cup b \subset V$.

Let F be a properly embedded orientable surface in the exterior X_K of a knot $K \subset \mathbb{S}^3$ and $F \times [-1,1]$ a thin regular neighborhood of F in X_K with $F = F \times \{0\}$. A surface S in X_K is said to *locally lie on one side of* F if $\partial S \subset F$, $F \cap \operatorname{int}(S) = \emptyset$, and either $S \cap (F \times [-1,0)) = \emptyset$ or $S \cap (F \times (0,1]) = \emptyset$; that is, near ∂S, S intersects only one side $F \times [0,1]$ or $F \times [-1,0]$ of $F \times [-1,1]$.

A companion annulus for a non-trivial circle $\gamma \subset F$ is an annulus A that locally lies on one side of F and is not parallel into F, with the circles ∂A isotopic to γ in F.

Examples of genus one knots $K_0 \subset \mathbb{S}^3$ with a once-punctured torus $F \subset X_K$ that contains a non-separating circle γ with companion annuli on either side of F can be constructed as follows. Let $L \subset \mathbb{S}^3$ be a cable knot with solid torus regular neighborhood $V \subset \mathbb{S}^3$ and essential annulus $B \subset X_L = \mathbb{S}^3 \setminus \operatorname{int}(V)$. Using a thin regular neighborhood $(\partial V) \times [0,1] \subset V$ of $\partial V = (\partial V) \times \{0\}$, extend B slightly into $\operatorname{int}(V)$ to an annulus $\widetilde{B} = B \cup ((\partial B) \times [0,1])$. Construct a pair of pants P embedded in V by suitably attaching a band $b \subset \operatorname{int}(V)$ to the annuli $(\partial B) \times [0,1] \subset V$ connecting the boundary circles $(\partial B) \times \{1\}$, in such a way that the circles $\partial_1 P \sqcup \partial_2 P = P \cap \partial V$ $(= \partial B)$, when oriented relative to P, end up with opposite orientations relative to ∂V, and the circle $K_0 = \partial_3 P$ is non-trivial in V (see Fig. 4). It follows that the knot K_0 is a satellite of L with winding number zero in V and $F = P \cup B$ is a once-punctured torus bounded by K_0; moreover, the core γ of B is a circle in F with companion annuli the closures of the components of $\partial V \setminus \partial B$, which lie on either side of F.

In fact, the argument of the next result shows that any such knot $K_0 \subset \mathbb{S}^3$ is obtained in this way.

Lemma 5.1. *Let $K \subset S^3$ be a genus one knot and $F \subset X_K$ a properly embedded incompressible once-punctured torus. If there is a non-trivial circle $\gamma \subset F$ which has companion annuli locally on either side of F then γ is non-separating in F and K is a satellite knot.*

Conversely, if an essential torus in X_K intersects F minimally in a nonempty collection of circles then there is a non-separating circle in F which has companion annuli locally on either side of F.

Proof. Let A and A' be companion annuli for $\gamma \subset F$ that locally lie on opposite sides of F. Without loss of generality, we may assume that A and A' have been isotoped so as to intersect minimally with the circles $\partial A = \partial A'$ cobounding an annular neighborhood $B \subset F$ of γ. Let $V, V' \subset X_K$ be the regions in X_K bounded away from ∂X_K by the closed tori $A \cup B, A' \cup B$, respectively, and let r denote the slope of γ in ∂V and $\partial V'$.

Suppose that γ is parallel to ∂F in F, and consider the companion annulus A of γ. Then A can be isotoped in X_K so that its boundary lies in ∂X_K, whence A becomes an essential annulus in X_K. It follows that either A is a cabling annulus for K, in which case $X_K(\partial F)$ has a lens space connected summand, or K is a composite knot with A a decomposing annulus having meridional boundary slope, neither of which is the case since ∂F is a longitude of K. Therefore γ is not parallel to ∂F and so γ is a non-separating circle in F.

Recall that $F \cap \text{int}(A) = \emptyset = F \cap \text{int}(A')$. If $A \cap A' \neq \emptyset$ then each component in a minimal intersection of A and A' is a core circle in A and A' and so it is possible to construct a closed surface S in X_K which intersects F transversely in the circle γ out of the annular components of $A \setminus A'$ and $A' \setminus A$. As γ is non-separating in F, it follows that S is a non-separating closed torus or a Klein bottle in $X_K \subset \mathbb{S}^3$, which is impossible.

Therefore $A \cap A' = \emptyset$ and so $V \cap V' = \emptyset$, hence $V \cup_B V'$ is a manifold with torus boundary which contains B as an essential annulus. Thus $V \cup_B V'$ is not a solid torus, so $V_L = \mathbb{S}^3 \setminus \text{int}(V \cup_B V')$ is a solid torus whose core L is a non-trivial knot in $\mathbb{S}^3$ with exterior $X_L = V \cup_B V'$ and $N(K) \subset V_L$. Since B is an essential annulus in X_L, the boundary slope $r \subset \partial X_L$ of B relative to the solid torus V_L is either meridional or integral.

Now, the surface $P = F \cap V_L$ is an incompressible pair of pants in V_L with $\partial_0 P = \partial F \subset \partial N(K)$ and $\partial_1 P \sqcup \partial_2 P \subset \partial V_L$ oppositely oriented circles of slope r relative to V_L. Thus K has zero winding number in V_L and is therefore not a core of V_L.

Suppose K is disjoint from some meridian disk D of V_L. If the slope r is meridional in V_L then the circle $\gamma \subset F$ bounds a disk in X_K and so F is not π_1-injective in X_K, contradicting the incompressibility of F. Therefore r is an integral slope in V_L, so if D and P are isotoped so as to intersect minimally then an outermost disk of $D \cap P \subset D$ boundary compresses P in V_L towards ∂V_L into an annulus whose boundary component in ∂V_L is a trivial circle; thus $\partial_0 P = \partial F$ bounds a disk in V_L, so K bounds a disk in V_L, contradicting the non-triviality of K in $\mathbb{S}^3$.

It follows that K is a non-trivial knot in the solid torus V_L, and hence that K is a satellite of the non-trivial knot L in $\mathbb{S}^3$.

Conversely, suppose that T is an essential torus in X_K which intersects F minimally in a nonempty collection of circles $T \cap F$. Then $T \cap F$ consists of at most two parallelism classes of circles in F: a class corresponding to the slope of some non-separating circle $\gamma \subset F$, and a class of circles parallel to ∂F.

Since T separates X_K, $T \cap F$ cannot consist of a single copy of the non-separating circle γ in F, hence the closure P of the component of $F \setminus T$ that contains ∂F is not equal to F.

Suppose that P is an annulus. If T bounds a solid torus $V \subset \mathbb{S}^3$ with $N(K) \subset V$ and $V_K = V \setminus \operatorname{int} N(K)$ is the exterior of K in V, then the annulus P is properly embedded in V_K and so K is a cable of the core of V with $\partial P \cap \partial N(K)$ the slope $m/1$ in $\partial N(K)$ of the cabling annulus of K, where necessarily $m \neq 0$, contradicting the fact that $\partial F = \partial P \cap \partial N(K)$ is the longitude of K.

Therefore P is not an annulus, which implies that all circles $T \cap F$ have slope γ in F, and hence that γ has companion annuli on both sides of F. □

5.2 Twice-punctured tori in X_K

In this section we assume that $K \subset S^3$ is a knot whose exterior X_K contains a properly embedded incompressible, separating, twice-punctured torus F with boundary slope $r \subset \partial X_K$, such that the closures F^B, F^W of the components of $X_K \setminus F$ are genus two handlebodies.

We consider the following auxiliary conditions:

(C1) there is a non-separating circle in F which is a power circle in F^B and F^W,

(C2) for some $\{*, **\} = \{B, W\}$ there are two mutually disjoint and non-isotopic non-separating circles in F which are power circles in F^* and primitive and coannular circles in F^{**}.

Lemma 5.2. *If the knot $K \subset \mathbb{S}^3$ is a satellite then either (C1) or (C2) holds and K is a satellite of a torus knot, and if (C2) holds then K is a genus one knot and the boundary slope r of F is the longitude of K.*

Proof. Let $K \subset \mathbb{S}^3$ be a satellite knot and $T \subset X_K$ an essential closed torus that bounds a solid torus $V \subset S^3$ with $K \subset N(K) \subset V$ whose core is a non-trivial knot with exterior $X = \mathbb{S}^3 \setminus \text{int}(V) \subset X_K$.

Isotope T so as to intersect F minimally. Since F^B and F^W are handlebodies, we must have that $T \cap F \neq \emptyset$. By the incompressibility of T and F and the minimality of $T \cap F$, each component of $T \cap F$ is a circle which is non-trivial in both T and F; thus for $* \in \{B, W\}$ each component of $T \cap F^*$ is an incompressible annulus in F^* which is not parallel into F.

Suppose A_1 is a component of, say, $T \cap F^B$, which is parallel in F^B into ∂F^B. By minimality of $T \cap F$, A_1 must be parallel in F^B to the annulus $F^B \cap N(K)$, that is, the circles $\partial_1 A_1, \partial_2 A_1$ must be parallel in F to the circles $\partial_1 F, \partial_2 F$. If A_2 is the component of $T \cap F^W$ with $\partial_1 A_2 = \partial_2 A_1$ then $\partial_1 A_2$ is neither a primitive nor power circle in F^W and so, by Lemmas 3.3 and 3.4, A_2 is parallel into ∂F^W. By minimality of $T \cap F$, it then follows that $T = A_1 \cup A_2$, hence that T is parallel in X_K to $\partial N(K)$, contradicting the hypothesis on T.

Therefore no annulus component of $T \cap F^*$ is parallel in F^* into ∂F^*, so again, by Lemmas 3.2, 3.3 and 3.4, in ∂F^* each component of $T \cap F$ is a non-separating primitive or power circle in F^*, and so in F the circles $T \cap F$ form at most two parallelism classes, neither one parallel to ∂F.

If some component of $T \cap F$ is a power circle in both F^B and F^W then (C1) holds. If some component γ_1 of $T \cap F$ is not a power circle in, say, F^B, then γ_1 is primitive in F^B and by Lemma 3.3 it has no companion annulus in F^B; hence the component A^B of $T \cap F^B$ with $\gamma_1 \subset \partial A^B$ must be a non-separating annulus in F^B. It follows that $T \cap F$ has two parallelism classes in F, represented by the circles $\partial A^B = \gamma_1 \sqcup \gamma_2 \subset F$.

Any component A^W of $T \cap F^W$ that is a non-separating annulus in F^W can be isotoped in F^W so that $\partial A^B = \partial A^W$, thus producing a closed Klein bottle or non-separating torus $A^B \cup_\partial A^W \subset X_K \subset \mathbb{S}^3$, which is impossible. Therefore $T \cap F^W$ is a union of a family of disjoint companion annuli for γ_1 and another family of disjoint companion annuli for γ_2. By Lemma 3.3(2), γ_1 and γ_2 must be power circles in F^B and so (C2) holds. Moreover, in this case the circles $\partial A^B = \gamma_1 \sqcup \gamma_2$ cut the surface F into two pairs of pants, hence the frontier of $N(A^B \cup F)$ in X_K consists of two disjoint once-punctured tori, each with boundary slope r, and so K is a genus one knot with longitudinal slope r. □

We remark that the converse of Lemma 5.2 holds, that is, if one of the conditions (C1) or (C2) is satisfied then K is a satellite knot, though we shall not make use of this fact.

Examples of knots $K \subset \mathbb{S}^3$ with such a twice-punctured incompressible torus $F \subset X_K$ satisfying condition (C1) or (C2) can be constructed, not exhaustively, as

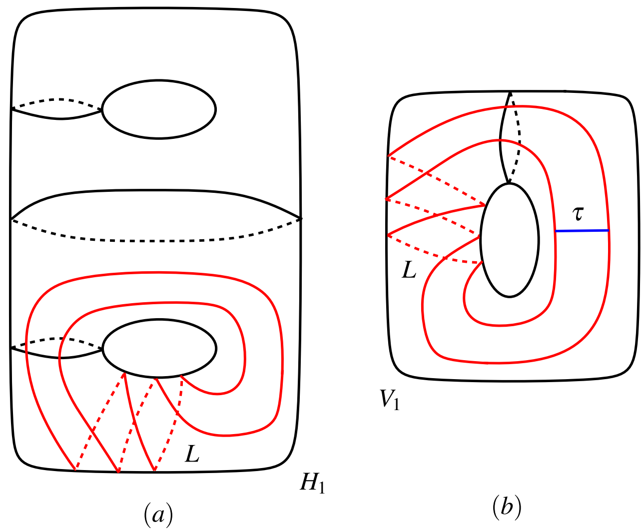

Figure 5: The (p_1, p_2)-torus knot L in H_1 and V_1 (with $(p_1, p_2) = (2,3)$).

follows. We begin by constructing two distinct genus two Heegaard splittings of $\mathbb{S}^3$ associated to any (p_1, p_2)-torus knot $L \subset \mathbb{S}^3$ with $p_1, p_2 \geq 2$. Fig. 5(a) shows a genus two handlebody H_1 standardly embedded in $\mathbb{S}^3$, which produces a Heegaard splitting $H_1 \cup H_2$ of $\mathbb{S}^3$, where the knot L is embedded in ∂H_1 in the 'bottom-half' solid torus part of H_1. Thus, for $i = 1,2$, L is power p_i circle in H_i.

Fig. 5(b) shows the knot L in the boundary of a solid torus V_1 which is part of a genus one Heegaard splitting $V_1 \cup V_2$ of $\mathbb{S}^3$. Let $N(L) \subset \mathbb{S}^3$ be a thin regular neighborhood of L, and for $i = 1,2$ let γ_i be a core of the annulus $V_i \cap \partial N(L) \subset \partial N(L)$, so that γ_i runs p_i times around V_i. As the arc $\tau \subset \partial V_1$ with endpoints in L shown in Fig. 5(b) is a tunnel for L, the genus two handlebody $H_1 = N(L \cup \tau) \subset \mathbb{S}^3$ is part of a Heegaard splitting $H_1 \cup H_2$ of $\mathbb{S}^3$. After a small isotopy if necessary, we may assume that the circles $\gamma_1 \sqcup \gamma_2$ lie in $\partial H_1 = \partial H_2$, whence γ_1, γ_2 are coannular primitive circles in H_1 while γ_i is a power p_i circle in H_2 for $i = 1,2$.

Clearly, if K is any circle embedded in $\partial H_1 \setminus L$ or $\partial H_1 \setminus (\gamma_1 \sqcup \gamma_2)$ which is neither a primitive nor a power circle in H_1 and H_2 (any 'sufficiently complicated' such embedding will do), then by Lemma 3.3(1) the knot K and the twice-punctured torus $F = (\partial H_1) \cap X_K$ satisfy condition (C1) or (C2), respectively.

6 Structure of pairs

We now take a closer look at the structure of pairs. We begin with a classification of pairs (H,J) of type (p_1,p_2), which include all simple and double pairs, in terms of the number of intersections of J with non-trivial disks in H. Each simple pair (H,J) is shown to have a distinguished core knot in H, and double pairs are shown to be obtained as a union of two simple pairs. Basic and primitive pairs are introduced in order to classify maximal pairs and to discuss properties of more general pairs (H,J), including the relationship between primitive, power, and Seifert circles in $\partial H \setminus J$. These properties will be used in later sections in the analysis of knot exteriors in S^3 that can be decomposed as a union of non-trivial pairs.

Lemma 6.1. *A pair (H,J) is of type (p_1,p_2) for some $p_1,p_2 \geq 1$ iff there is a disk in H which intersects J minimally in 4 points.*

Proof. Let (H,J) be a pair of type (p_1,p_2). By construction, the pair is obtained by attaching solid tori to the genus two handlebody $F \times I$ shown in Fig. 1 along the circles $\gamma_1,\gamma_2 \subset F \times I$; clearly the waist disk $D_w \subset F \times I$ shown in Fig. 1 lies in H and intersects J minimally in 4 points.

Conversely, suppose that (H,J) is a pair with $\partial H = T_1 \cup_J T_2$ and $E \subset H$ is disk which intersects J minimally in 4 points. Then, for $i = 1,2$, $T_i \cap \partial E$ consists of 2 arcs such that either (1) for $i = 1,2$, the arcs $T_i \cap \partial E$ are parallel in T_i, in which case E is a separating disk, or (2) for some $\{i,j\} = \{1,2\}$, the arcs $T_i \cap \partial E$ are parallel in T_i and the arcs $T_j \cap \partial E$ are non-parallel in T_j, in which case E is a non-separating disk. Notice that, by connectedness of ∂E, the case where the arcs $T_i \cap E$ are not parallel in T_i for $i = 1,2$ does not occur.

In case (1) E is a waist disk for H. Let $\gamma_i \subset T_i$ be the unique circle in T_i which is disjoint from the arcs $T_i \cap \partial E$. Then the circles γ_1,γ_2 are separated in H by E and so, for each $i = 1,2$, γ_i is a power p_i circle for some $p_i \geq 1$, which implies that (H,J) is a pair of type (p_1,p_2).

In case (2) we may assume that $(i,j) = (1,2)$, and Fig. 6 shows the triple $(\partial H,J,\partial E)$ up to homeomorphism. Since the arcs $T_2 \cap \partial E$ are not parallel in T_2, there is a unique circle $\gamma_2 \subset T_2$ which intersects the arcs $T_2 \cap \partial E$ each minimally in one point with algebraic intersection number $\gamma_2 \cdot \partial E = \pm 2$ (see Fig. 6). If E' is a disjoint parallel copy of E and α is any arc component of $\gamma_2 \setminus (\partial E \cup \partial E')$ not in the parallelism region between E and E' then $E_0 = \mathrm{fr}\, N(E \cup \alpha \cup E')$ is a waist disk of H which can be isotoped so as to intersect J minimally in 4 points and be disjoint from $E \cup \gamma_2$. Therefore, by case (1) the pair (H,J) is of type (p_1,p_2) for some $p_1,p_2 \geq 1$. In fact, since E and D form a complete disk system for H, it follows that γ_2 is a power 2 circle in H and hence that $p_2 = 2$. □

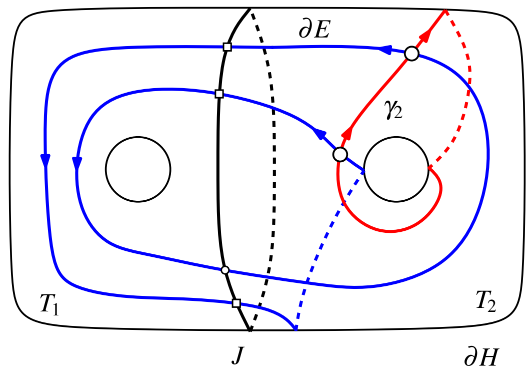

Figure 6: The unique circle $\gamma_2 \subset T_2$ with $\Delta(\gamma_2, \partial E) = |\gamma_2 \cdot \partial E| = 2$.

6.1 Cores of simple pairs

The next result classifies simple pairs via power circles and summarizes some of their properties.

Lemma 6.2. *Let (H,J) be a pair with $\partial H = T_1 \cup_J T_2$. Then (H,J) is a simple pair of type $(1,p)$ for some $p \geq 2$ iff the pair (H,J) is minimal and there is a circle in T_1 or T_2 which is a power p circle in H, in which case*

1. *there are power p circles $\omega_i \subset T_i$ which are coannular in H and such that $\partial H \setminus (\omega_1 \sqcup \omega_2)$ compresses along a non-separating disk $D \subset H$ that intersects J minimally in 2 points.*

2. *any power circle in T_i is isotopic to ω_i,*

3. *any disk in H which intersects J minimally in 2 points is isotopic to D,*

4. *$H(J) = \mathbb{A}^2(p)$ with singular fiber of index p represented by the core K of the solid torus $H|D$ and regular fibers the circles $\omega_i \subset \widehat{T_i} \subset \partial H(J)$; moreover, if (H,J) is a simple pair of type $(0,1;a,p)$ then there are essential annuli $A_1, A_2 \subset H \setminus N(K)$ with $\partial_1 A_i = \omega_i$ and $\partial_2 A_i \subset \partial N(K)$ a circle of type (a,p) in $N(K)$ (see Fig. 7),*

5. *if a non-separating circle $\alpha \subset T_i$ intersects ω_i and D minimally then $|\alpha \cap \omega_i| = |\alpha \cap D|$; in particular, $\alpha \subset T_i$ is primitive in H iff $|\alpha \cap \omega_i| = 1 = |\alpha \cap D|$,*

and if $q = |\alpha \cap \omega_i| = |\alpha \cap D|$ *then*

$$H(\alpha) = H(J)(\alpha) = \begin{cases} (\mathbb{S}^1 \times \mathbb{D}^2)\#L_p & q = 0 \\ \mathbb{S}^1 \times \mathbb{D}^2 & q = 1 \\ \mathbb{D}^2(p,q) & q \geq 2. \end{cases}$$

Proof. Suppose that (H,J) is a simple pair of type $(1,p)$ for some $p \geq 2$. Thus $H = (F \times I) \cup V$ for some once-punctured torus F, where $J \subset \partial H$ is the core of the annulus $(\partial F) \times I$ and V is a solid torus glued to $F \times I$ along an annular neighborhood of some non-separating circle $\gamma \subset F \times \{0\}$, such that γ runs p times around V. If $\gamma_0 \subset F \times \{0\} \setminus V$ is a circle parallel to γ in $F \times \{0\}$ and $\delta_0 \subset F \times \{0\}$ is an essential arc in $F \times \{0\}$ disjoint from $V \sqcup \gamma_0$ then the annulus $B = \gamma_0 \times I \subset F \times I$ is properly embedded in H with boundary a pair of coannular power p circles $\omega_1 = \partial_1 B = \gamma_0 \times \{0\} \subset F \times \{0\} \subset T_1$ and $\omega_2 = \partial_2 B = \gamma_0 \times \{1\} \subset F \times \{1\} \subset T_2$ in H, and $D = \delta_0 \times I$ is a non-separating disk properly embedded in H which intersects J minimally in two points and is disjoint from $\omega_1 \sqcup \omega_2$. That ω_i is the only power circle in T_i follows from Lemma 3.6, while by Lemma 3.3(1), the disk $D \subset H$ is the unique compression disk for $\partial H \setminus \omega_1$; thus (1), (2) and (3) hold.

As γ is a primitive circle in $F \times I$, γ is also primitive in the solid torus $F \times I|D$ and so the core of V and the core K of the solid torus $H|D = (F \times I|D) \cup_\gamma V$ are isotopic in $H|D \subset H$. From the identity

$$H(J) = (F \times I)(J) \cup_\gamma V = (\widehat{F} \times I) \cup_\gamma V$$

it follows that the manifold $H(J)$ is a Seifert fiber space $\mathbb{A}^2(p)$ over the annulus with singular fiber $K \subset H$ of index p and regular fibers $\omega_i \subset \widehat{T}_i$ and so (4) holds.

Finally, let $\alpha \subset T_1$ be any non-separating circle, and consider the arc $\delta_1 = T_1 \cap \partial D \subset D$. After isotoping α in T_1 so as to intersect $\omega_1 \cup \delta_1 \subset T_1$ minimally we must have $q = |\alpha \cap \omega_1| = |\alpha \cap \delta_1| = |\alpha \cap D|$. Since J bounds a disk in $H(\alpha)$ we have the identity $H(\alpha) = H(J)(\alpha) = \mathbb{A}^2(p)(\alpha)$; therefore,

$$\begin{aligned} \alpha \text{ is primitive in } H &\iff H(\alpha) = \mathbb{A}^2(p)(\alpha) \text{ is a solid torus} \\ &\iff |\alpha \cap \omega_1| = 1, \end{aligned}$$

and the rest of (5) follows in a similar way. □

- We will call the knot $K \subset H$ in Lemma 6.2(4) the *core of the simple pair* (H,J), and say that K and the pair (H,J) have *index* $p \geq 2$.

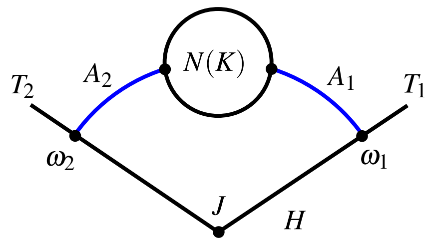

Figure 7: The core knot K and power circles $\omega_1 \subset T_1$ and $\omega_2 \subset T_2$ of a simple pair (H,J).

Lemma 6.3. *Let (H,J) be a simple pair with $\partial H = T_1 \cup_J T_2$, core knot $K \subset H$, power circles $\omega_1 \subset T_1$ and $\omega_2 \subset T_2$, and incompressible annuli $A_1, A_2 \subset H \setminus \mathrm{int} N(K)$ as shown in Fig. 7. Then the solid torus $V_1 = N(A_1) \cup N(K) \subset H$ has core K and is the companion solid torus of the power circle ω_1, and there is a homeomorphism*

$$H' = cl[H \setminus V_1] = cl[H \setminus (N(A_1) \cup N(K))] \approx T_2 \times I$$

such that $T_2 \subset H'$ corresponds to $T_2 \times \{0\} \subset T_2 \times I$ and the circle $A_2 \cap N(K) \subset \partial H'$ to $\omega_2 \times \{1\} \subset T_2 \times \{1\}$.

In particular, if (H^*, J^*) is a pair with $\partial H^* = T_1^* \cup_{J^*} T_2^*$ and $M = H \cup_{T_1 = T_1^*} H^*$ then M is a handlebody iff $\omega_1 \subset T_1 = T_1^*$ is primitive in H^*.

Proof. $V_1 = N(A_1) \cup N(K) \subset H$ is indeed a solid torus with core K, with the power $p \geq 2$ circle ω_1 running p times around V_1 by Lemma 6.2(4). Therefore $\mathrm{fr}(N(A_1) \cup N(K))$ is a companion annulus for ω_1 in H with companion solid torus V_1, both of which are unique up to isotopy in H by Lemma 3.3, and so the first part follows from the definition of a simple pair. As the homeomorphism $H' = \mathrm{cl}[H \setminus (N(A_1) \cup N(K))] \approx T_2 \times I$ induces a homeomorphism $M \approx H^* \cup_{\omega_1} V_1$, the second part now follows from Lemma 3.5(1). □

Proof. Since $\mathrm{fr}(N(A_1) \cup N(K))$ is a companion annulus for ω_1 in H with companion solid torus $V_1 = N(A_1) \cup N(K)$, both of which are unique up to isotopy in H by Lemma 3.3, the first part follows from the definition of a simple pair. As the homeomorphism $H' = \mathrm{cl}[H \setminus (N(A_1) \cup N(K))] \approx T_2 \times I$ induces a homeomorphism $M \approx H^* \cup_{\omega_1} V_1$, the second part now follows from Lemma 3.5(1). □

Lemma 6.4. *Let (H,J) be a simple pair with $\partial H = T_1 \cup_J T_2$, core knot $K \subset H$ of index $p \geq 2$, and power circles $\omega_1 \subset T_1$ and $\omega_2 \subset T_2$. Denote by $XH_K = H \setminus$*

$\operatorname{int} N(K)$ the exterior of K in H, and by r the slope in $\partial N(K)$ corresponding to the circles $\partial_2 A_i$ (see Fig. 7).

If $\alpha_1 \subset T_1$ and $\alpha_2 \subset T_2$ are primitive circles in H then there is a unique slope $s \subset \partial N(K)$ such that the circles α_1 and α_2 are coannular in the handlebody $XH_K(s)$, and the following conditions hold:

1. *$\Delta(s,r) = 1$ and the pair $(XH_K(s), J)$ is trivial,*
2. *there is a unique circle $s' \subset \partial H \setminus (\alpha_1 \sqcup \alpha_2)$ which cobounds an annulus $\mathscr{A}$ in XH_K with $s \subset \partial N(K)$; s' intersects each circle $\omega_1, \omega_2 \subset \partial H$ minimally in one point,*
3. *the circles $\alpha_1 \sqcup \alpha_2 \subset \partial H$ and $s \subset \partial N(K)$ cobound a pair of pants $\mathscr{P}$ in XH_K disjoint from the annulus $\mathscr{A}$,*
4. *the slope s is integral in $N(K)$ iff α_1, α_2 are basic circles in H iff s' is a primitive circle in H, in which case each circle α_1, α_2 runs once around the solid torus $H(s')$.*

Proof. By Lemma 6.2 there is a unique disk $D \subset H$ which intersects J minimally in two points, is disjoint from $\omega_1 \sqcup \omega_2$, and intersects each primitive circle α_1, α_2 minimally in one point. Thus the frontier D_w of a thin regular neighborhood of $\alpha_1 \cup D$ is a waist disk of H which minimally intersects J in 4 points and the circle α_2 in 2 points.

The waist disk D_w separates H into two solid tori V, V' with $V \cap V' = D_w$ and D a meridian disk of V. Since the solid tori $H|D$ and V' are isotopic in H, the core knot K of the pair (H,J) can be identified with the core circle of V'. Therefore the exterior $V'_K = V' \setminus \operatorname{int} N(K)$ of K in V' is a product of the form $(\partial N(K)) \times I$ and $V'_K(s)$ is a solid torus with $\partial V'_K(s) = \partial V'$ for each slope $s \subset \partial V'$. In particular we have that $XH_K(s) = V \cup_{D_w} V'_K(s)$ is a handlebody, and each slope $s \subset \partial N(K)$ cobounds an annulus $\mathscr{A}$ in V'_K with a unique slope $s' \subset \partial V' \setminus D_w$, so that s' bounds a meridian disk D'_s in the solid torus $V'_K(s)$.

We also have that $\omega_2 \subset \partial V' \setminus D_w$ and that $t_2 = \alpha_2 \cap \partial V'$ is a single arc which, by Lemma 6.2(5), intersects ω_2 minimally in one point. The situation is represented in Fig. 8(a), where for simplicity we have used $p = 2$ and a specific primitive circle $\alpha_2 \subset T_2$; the circle α_1 is not shown in this figure.

For any slope $s \subset \partial N(K)$, as D'_s is disjoint from α_1, by Lemmas 3.3(1)(b) and 3.4 the circles α_1, α_2 are coannular in $XH_K(s)$ iff D'_s is disjoint from α_2, that is, iff the circle $s' = \partial D'_s$ is disjoint from the arc t_2. Since, up to isotopy, there is a unique such circle $s' \subset \partial V' \setminus D_w$, namely the circle obtained as the union of t_2 and a component of $\partial D_w \setminus t_2$, it follows that there is a unique slope $s \subset \partial N(K)$ such

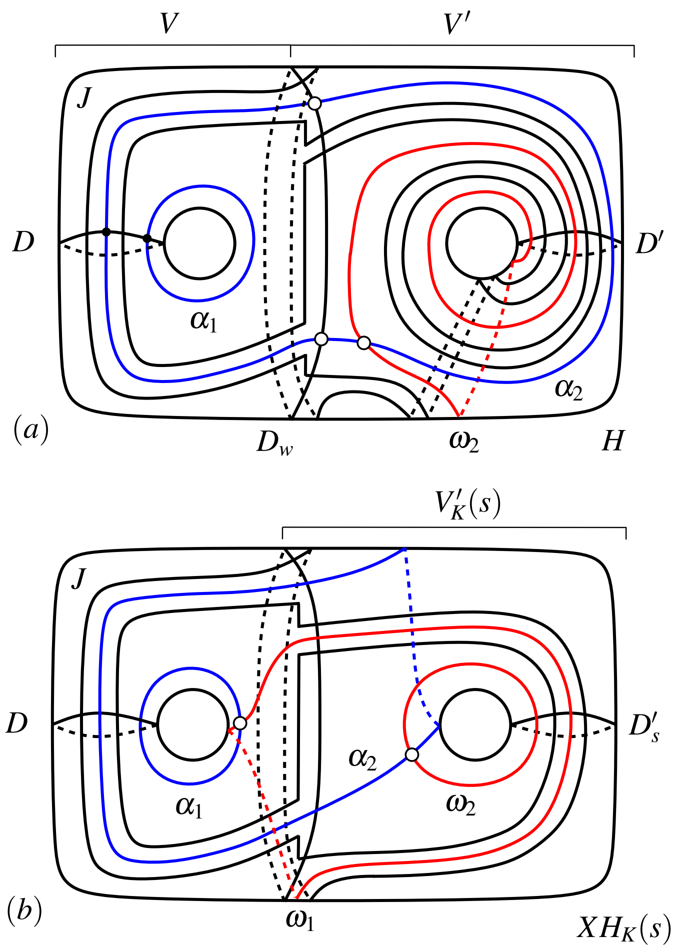

Figure 8: The circles $\alpha_1, \alpha_2, \omega_1, \omega_2$ in ∂H and $\partial XH_K(s)$.

that α_1 and α_2 are coannular in $XH_K(s)$, in which case $\Delta(s,r)=1$ since $|s'\cap\omega_2| = |t_2\cap\omega_2| = 1$, and the corresponding meridian disk $D'_s \subset V'_K(s)$ is disjoint from α_2 and intersects ω_2 minimally in one point; thus D'_s intersects $J=\partial N(\alpha_1\cup\omega_2)$ minimally in two points and so the pair $(XH_K(s),J)$ is of type $(1,1)$, that is, trivial. Moreover, the circles α_1,α_2,s' are necessarily mutually non-parallel in ∂H and hence separate ∂H into two pairs of pants, so α_1,α_2 and the slope s cobound a pair of pants $\mathscr{P}$ in XH_K disjoint from $\mathscr{A}$.

Finally, let D' be a meridian disk of V' which is disjoint from $D_w\subset\partial V'$ and intersects α_2 minimally, and let $x,y\subset H$ be circles dual to D,D', respectively, which represent a basis for $\pi_1(H)$. Then there is a nonzero integer m such that, in $\pi_1(H)=\langle x,y \mid - \rangle$, $\alpha_1 = x$ and $\alpha_2 = xy^m$. It follows from the above construction (see Fig. 8(a)) that $s'=y^m$ in $\pi_1(H)$, hence that s runs $|m|$ times around $N(K)$, and hence that

$$\begin{aligned}\text{the slope } s \text{ is integral} &\iff |m|=1\\ &\iff \alpha_1,\alpha_2 \text{ are basic circles in } H\\ &\iff s' \text{ is primitive in } H,\end{aligned}$$

in which case $H(s')$ is a solid torus and the circles α_1,α_2 run once around $H(s')$. □

6.2 Basic simple pairs and Seifert circles

A pair (H,J) with $\partial H = T_1\cup_J T_2$ is a *basic pair* if there are circles $\alpha_1\subset T_1$ and $\alpha_2\subset T_2$ which are basic in H. Any trivial pair is basic, and the next result classifies the simple pairs that are basic. The construction of general basic pairs will be discussed in Remark 7.7.

Lemma 6.5. *Let (H,J) be a simple pair of type $(0,1;a,p)$ with $\partial H = T_1\cap_J T_2$ and unique meridian disk $D\subset H$ with $|D\cap J|=2$. Then (H,J) is a basic pair iff $a\equiv\pm1$* mod *p, in which case if $\alpha_1\subset T_1$ is any primitive circle in H then*

1. *there is a circle $\alpha_2\subset T_2$ such that α_1,α_2 are basic circles in H,*

2. *up to isotopy in T_2, the circle $\alpha_2\subset T_2$ is unique if $p\geq 3$, and there are exactly 2 such circles if $p=2$,*

3. *for each pair of basic circles $\alpha_1\subset T_1$ and $\alpha_2\subset T_2$ there is a unique complete disk system D',D'' of H disjoint from D such that $|D''\cap\alpha_1| = 1 = |D'\cap\alpha_2|$, $|D''\cap\alpha_2| = 0 = |D'\cap\alpha_1|$; moreover, the 7-tuple $(H,J,D,D',D'',\alpha_1,\alpha_2)$ is homeomorphic to the one shown in Fig. 9(b) (where $p=2$ is used for simplicity).*

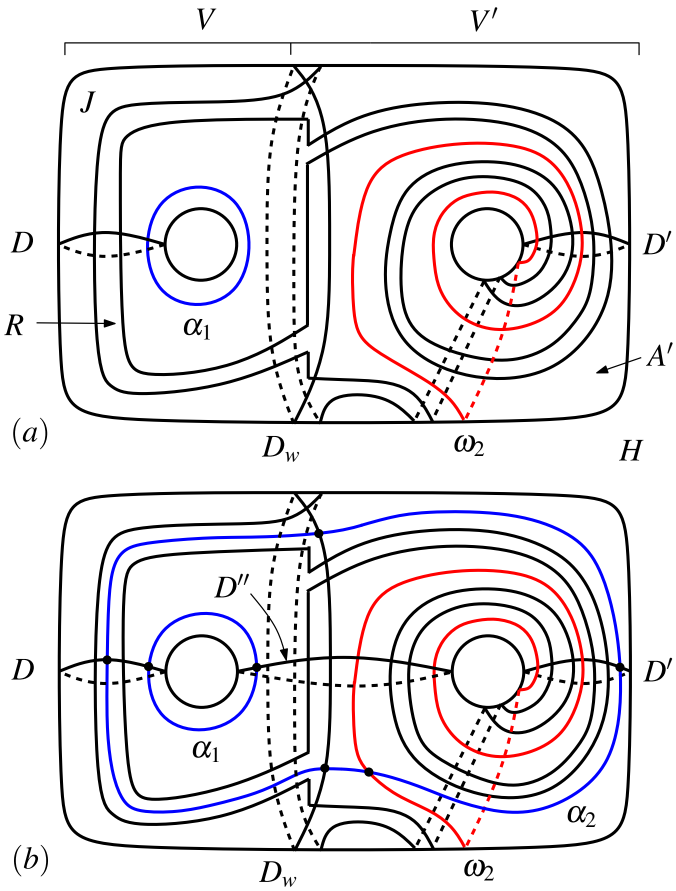

Figure 9: A basic simple pair (H,J) of index $p=2$.

Proof. By Lemmas 3.11 and 6.2(3),(5) there is a unique disk $D \subset H$ which intersects J minimally in two points such that a circle in T_1 or T_2 is primitive in H iff it intersects D minimally in one point.

If $\alpha_1 \subset T_1$ is any primitive circle in H then the frontier D_w of a regular neighborhood of $\alpha_1 \cup D$ is a waist disk of H which intersects J minimally in 4 points. Therefore the 5-tuple $(\partial H, J, \partial D, \partial D_w, \alpha_1)$ is homeomorphic to the 5-tuple $(\partial(F \times I), J, D_1, D_w, \gamma_1)$ of Fig. 1, which implies that the 5-tuple (H, J, D, D_w, α_1) is homeomorphic to the one shown in Fig. 9(a) (where $p=2$ is used for simplicity).

We construct a circle $\gamma \subset T_2$ which intersects ω_2 minimally in one point as follows. The waist disk D_w separates H into two solid tori V, V' with $V \cap V' = D_w$ and meridian disks $D \subset V \setminus D_w$ and $D' \subset V' \setminus D_w$, such that $\omega_2 \subset \partial V'$ represents a circle of type (a,p) in V', $R = V \cap T_2$ is a rectangle intersected by one arc of ∂D, and $A' = V' \cap T_2$ is an annular neighborhood of ω_2 in T_2.

Let $t \subset A'$ be any properly embedded arc with endpoints in ∂D_w which intersects ω_2 minimally in one point. Then t along with an arc of ∂D_w produces a closed circle $\widehat{t} \subset \partial V'$ of type (c,q) in V' for some integers c,q such that $|q| = |D' \cap t|$ and $aq - pc = \pm 1$; thus $a \equiv \pm q^{-1} \mod p$.

The union of the arc t with a core arc in the rectangle $R = V \cap T_2$ which intersects $\partial D \cap R$ minimally in one point produces the desired circle γ (see Fig. 9(b)). Since γ and ω_2 form a basis for the integral first homology group of T_2, if $\alpha_2 \subset T_2$ is any circle which intersects ω_2 minimally in one point then, homologically, the identity $\alpha_2 = \gamma + n\omega_2$ holds in T_2 for some integer n.

Therefore, if x and y represent the basis of $\pi_1(H)$ dual to the complete disk system $D, D' \subset H$, respectively, then, in $\pi_1(H) = \langle x, y \mid - \rangle$, under some orientation scheme, we can write $\alpha_1 = x$ and $\alpha_2 = x \cdot (y^p)^m \cdot y^q = x \cdot y^{mp+q}$ for some $m \in \mathbb{Z}$. Hence α_1, α_2 are basic circles in H iff $mp + q = \pm 1$, so $q \equiv \pm 1 \mod p$, and so $a \equiv \pm q^{-1} = \pm 1 \mod p$.

Now, there is at most one solution m for each equation $mp + q = \pm 1$, and there are integers m_1, m_2 with $m_1 p + q = 1$ and $m_2 p + q = -1$ iff $p = 2$ and q is odd, in which case $m_1 - m_2 = 1$. Hence $\alpha_2 = \gamma + m\omega_2$ is unique up to isotopy if $p \geq 3$, and there are two such circles α_2 if $p = 2$.

Since D' is disjoint from α_1, if α_1 and α_2 are basic circles in H then by Lemma 3.3(1)(b) D' is the unique non-separating compression disk of $\partial H \setminus \alpha_1$, while $|D' \cap \alpha_2| = |mp + q| = 1$. As $|D \cap \alpha_1| = 1 = |D \cap \alpha_2|$, cutting ∂H along $\partial D \cup \alpha_2 \cup \partial D'$ produces an annulus $A \subset \partial H$ which intersects α_1 minimally in one spanning arc. Thus the core of A is a circle in ∂H disjoint from $D \cup \alpha_2 \cup D'$ that bounds a non-separating disk D'' in H, hence by Lemma 3.3(1)(b) D'' must be the unique compression disk for $\partial H \setminus \alpha_2$. Therefore the 7-tuple $(H, J, D, D', D'', \alpha_1, \alpha_2)$ is homeomorphic to the one shown in Fig. 9(b) (where $p = 2$ for simplicity and one of the two possible circles α_2 is shown).

Conversely, if $a \equiv \pm 1 \mod p$ then the 4-tuple (H, J, D, D_w) is homeomorphic to the one shown in Fig. 9(b) and so a circle $\alpha_2 \subset T_2$ representing $xy^{\pm 1}$ in $\pi_1(H)$ can be easily constructed, in which case $\alpha_1 = x$ and $\alpha_2 = xy^{\pm 1}$ are basic circles in H. □

A circle $\alpha \subset \partial H$ in a genus two handlebody H is a *Seifert circle* if the manifold $H(\alpha)$ is a Seifert fiber space of the form $\mathbb{D}^2(*,*)$.

In the following result, we use the structure of the annuli obtained by 2-handle addition on a 3-manifold given in [4, Theorem 1] in order to characterize the Seifert circles $\alpha \subset \partial H$ in terms of properties of the surface $\partial H \setminus \alpha$ or the pair (H, α). Its statement uses the concept of a *primitive pair* (H, J), a non-trivial pair that contains a non-separating annulus whose boundary components are primitive circles in H separated by J; the properties of primitive pairs will be developed in §6.4.

Lemma 6.6. *Let H be a genus two handlebody and $\alpha \subset \partial H$ a circle such that $\partial H \setminus \alpha$ is incompressible in H. If $H(\alpha)$ contains an essential annulus A' with $\partial A' \subset \partial H \setminus \alpha$ then one of the following conditions holds:*

1. *there is a circle in $\partial H \setminus \alpha$ which is a power circle in H (necessarily, its companion annulus is essential in $H(\alpha)$),*
2. *α separates ∂H and the pair (H, α) is trivial or primitive,*
3. *α is non-separating in ∂H and $H(\alpha \sqcup \partial A') = L_p$ for some $p \neq 1$.*

Proof. If $H(\alpha)$ contains an essential annulus A' then, by [4, Theorem 1] and Remark (d) after its statement, there is an essential annulus $A \subset H(\alpha)$ satisfying condition (a) or (b) of that theorem and whose boundary is parallel in ∂H to one or both of the components of $\partial A'$.

Suppose first that part (a) of [4, Theorem 1] holds, that is, the annulus A lies in H with $\partial A \subset H \setminus \alpha$, which by [4, Theorem 1] is the case if α separates ∂H. Necessarily A is incompressible and not boundary parallel in H, so by Lemma 3.2 each component of ∂A is a non-separating circle in ∂H, and by Lemmas 3.3(2) and 3.4 both components of ∂A are primitive or both are power circles in H. In the latter case (1) holds, so assume that the circles ∂A are primitive in H. Since A is not boundary parallel in H, by Lemma 3.3(2) A must be a non-separating annulus and so the circles $\alpha, \partial_1 A, \partial_2 A$ are mutually disjoint and non-parallel in ∂H, and $H(\partial_1 A)$ is a solid torus with meridian circle $\partial_2 A$. If $\alpha \subset \partial H$ is a separating circle then by definition the pair (H, α) is either trivial or primitive, so (2) holds, while if α is non-separating then the circles $\partial_2 A$ and α are parallel in $\partial H(\partial_1 A)$ and hence $H(\alpha \sqcup \partial A') = H(\partial A) = \mathbb{S}^1 \times \mathbb{S}^2 = L_0$, so (3) holds.

Suppose now that part (b) of [4, Theorem 1] holds but not part (a), so that α is a non-separating circle in ∂H and no circle in $\partial H \setminus \alpha$ is a power circle in H. By Remark (b) after the statement of [4, Theorem 1], there is an incompressible, non-boundary parallel pair of pants $P \subset H$ with two boundary components $\partial_1 P, \partial_2 P \subset \partial H \setminus \alpha$ which are non-separating and mutually parallel, and a third boundary component $\partial_3 P \subset \partial H \setminus \alpha$ which separates $\partial_1 P \sqcup \partial_2 P$ from α, such that the surface $\widehat{P} \subset H(\alpha)$ obtained by capping off $\partial_3 P$ with a disk in $H(\alpha)$ is an essential separating annulus with the same boundary slope as A'. Moreover, $\widehat{P}$ separates $H(\alpha)$ into two components N and T, where T is a solid torus such that if $\tau \subset H(\alpha)$ is the cocore of the 2-handle attached to H along α, then τ can be slid over itself to form the union of an arc τ_2 and a core τ_1 of T, where $\tau_2 \cap T$ is a straight arc in T from ∂T to τ_1; the situation is represented in Fig. 10. Therefore

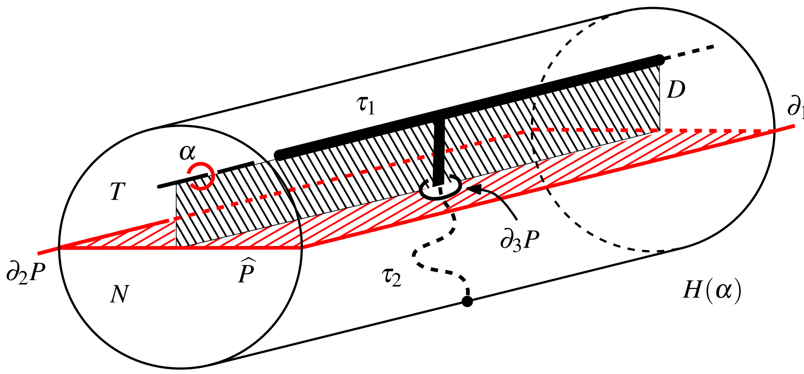

Figure 10: The manifold $H(\alpha)$ and the disk $D \subset H_2 \subset T$ ($p = 1$).

$H = \mathrm{cl}\,[H(\alpha) \setminus N(\tau_1 \cup \tau_2)]$, and in ∂H the meridian circle of $N(\tau_1) \subset T$ is isotopic to α while the meridian circle of $N(\tau_2)$ is isotopic to $\partial_3 P$.

The circle $\partial_3 P$ separates ∂H into two once-punctured tori T_1, T_2, with $\partial_1 P \sqcup \partial_2 P \subset T_1$, $\alpha \subset T_2$, and $\partial T_1 = \partial_3 P = \partial T_2$, while the incompressible surface P separates H into two genus two handlebodies H_1, H_2 (see [19, Lemma 2.3]), where the notation is chosen so that $\alpha \subset T_2 \subset \partial H_2$ and hence $H_2(\alpha) = T$.

Since the annulus $\widehat{P}$ is not boundary parallel in $H(\alpha)$ and $H(\alpha) = H_1(\partial_3 P) \cup_{\widehat{P}} H_2(\alpha)$, $\widehat{P}$ must run $p \geq 2$ times around the solid torus $T = H_2(\alpha)$. Thus there is a disk D properly embedded in H_2 which is disjoint from $\partial_1 P \sqcup \partial_2 P$ and intersects P in one arc, $\partial_3 P$ minimally in two points, and α minimally and coherently in p points; the disk D is shown in Fig. 10 (in the case $p = 1$ for simplicity).

Boundary compressing P in H along D produces two non-separating annuli $B_1, B_2 \subset H$, where $\partial_1 B_1 = \partial_1 P$, $\partial_1 B_2 = \partial_2 P$, and $\partial_2 B_1, \partial_2 B_2$ are parallel circles in $T_2 \subset \partial H$ with $\Delta(\alpha, \partial_2 B_1) = p = \Delta(\alpha, \partial_2 B_2)$. By Lemma 3.4 and our hypothesis on $\partial H \setminus \alpha$ not containing any power circles in H, $\partial_1 B_1$ and $\partial_1 B_2$ are primitive circles in H. Therefore the pair $(H, \partial_3 P)$ is primitive and $H(\partial_1 B_1)$ is a solid torus with meridian disk $\widehat{B}_1$ such that $\Delta(\alpha, \partial \widehat{B}_1) = \Delta(\alpha, \partial_2 B_1) = p$, whence $H(\alpha \sqcup \partial A') = H(\alpha \sqcup \partial_1 B_1) = L_p$ so (3) holds. □

Lemma 6.7. *Let H be a genus two handlebody and $\alpha \subset \partial H$ a non-separating circle. Then α is a Seifert circle in H iff the surface $\partial H \setminus \alpha$ is incompressible in H and contains a power circle β with companion annulus $B \subset H$ such that (1) α is a primitive circle in the handlebody $H_B \subset H|B$, in which case (2) β is a regular fiber of $H(\alpha)$, and any power circle in $\partial H \setminus \alpha$ satisfies (1) and (2).*

Proof. If α is a Seifert circle in H then by Lemma 3.3 and the 2-handle addition theorem the surface $\partial H \setminus \alpha_1 \subset H$ is necessarily incompressible and contains a power $p \geq 2$ circle by Lemma 6.6 applied to the unique separating essential annulus A' in $H(\alpha) = \mathbb{D}^2(*,*)$.

Let $B \subset H$ and $V_B \subset H$ be the companion annulus and solid torus of β, respectively. From the identity $H(\alpha) = H_B(\alpha) \cup_B V_B = \mathbb{D}^2(*,*)$ it follows that the annulus B is essential in $H_B(\alpha)$ and hence that $H_B(\alpha)$ is a solid torus. Therefore α is a primitive circle in H_B and the circles ∂B, and hence β, are regular fibers of $H(\alpha) = \mathbb{D}^2(*,*)$. The converse holds by a similar argument. □

6.3 Double and maximal pairs

Lemma 6.8. *Let (H,J) be a pair with $\partial H = T_1 \cup_J T_2$.*

1. *Suppose that $\omega_1 \subset T_1$ and $\omega_2 \subset T_2$ are, respectively, power p_1 and p_2 circles in H that induce disjoint once-punctured tori T_1' and T_2' in H with boundary slope J. Then $T_1' \sqcup T_2'$ cut H into 3 genus two handlebodies H_0, H_1, H_2 as shown in Fig. 11(a), such that*
   - *(a) (H_1,J) and (H_2,J) are simple pairs of types $(1,p_1)$ and $(1,p_2)$, respectively,*
   - *(b) (H_0,J) is a basic pair; specifically, the power circles $\omega_1' \subset T_1'$ in H_1 and $\omega_2' \subset T_2'$ in H_2 are basic circles in H_0, with ω_1' and ω_2' primitive circles in $H_0 \cup H_2$ and $H_0 \cup H_1$, respectively,*
   - *(c) if (H_0,J) is a simple pair with power circles $\gamma_1 \subset T_1'$ and $\gamma_2 \subset T_2'$ then, for each $i \in \{1,2\}$, $\Delta(\omega_i', \gamma_i) = 1$ and γ_i is a primitive circle in H_i,*
   - *(d) if the pair (H_0,J) is non-trivial then any non-separating circle $\alpha_1 \subset T_1$ which is not isotopic to ω_1 in T_1 is neither a primitive nor a power circle in H; in particular, the surface $\partial H \setminus \alpha_1$ is incompressible in H and the manifold $H(\alpha_1)$ is irreducible with incompressible boundary, with α_1 a Seifert circle in H iff α_1 is primitive in $H_0 \cup H_1$.*

2. *(H,J) is a double pair of type (p_1,p_2) iff there is a once-punctured torus $T \subset H$ with $\partial T = J$ that separates H into simple pairs (H_1,J) and (H_2,J) of types $(1,p_1)$ and $(1,p_2)$, respectively (see Fig. 11(b)), in which case*
   - *(a) any once-punctured torus in H bounded by J is parallel to T, T_1, or T_2,*
   - *(b) if $\omega_1' \subset T \subset H_1$ and $\omega_2' \subset T \subset H_2$ are the power circles in H_1 and H_2 then $\Delta(\omega_1', \omega_2') = 1$, ω_1' is a primitive circle in H_2, and ω_2' is a primitive circle in H_1,*
   - *(c) if $\alpha_1 \subset T_1$ is any non-separating circle which intersects ω_1 minimally in q points, then*

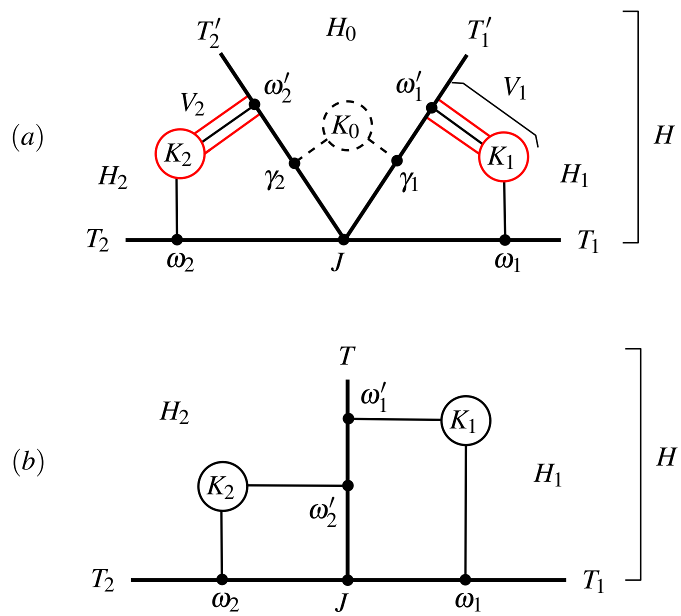

Figure 11: Once-punctured tori in a genus two handlebody H.

$$H(\alpha_1) = H_2(J) \cup_{\widehat{T}} H_1(\alpha_1)$$
$$= \begin{cases} (\mathbb{S}^1 \times \mathbb{D}^2)\# L_{p_1} & q = 0 \\ \mathbb{S}^1 \times \mathbb{D}^2 & q = 1 \textit{ and } \alpha_1 \textit{ is primitive in } H \\ \mathbb{D}^2(p_2, r) \textit{ for some } r \geq 2 & q = 1 \textit{ and } \alpha_1 \textit{ is not primitive in } H \\ \mathbb{A}^2(p_2) \cup_{\widehat{T}} \mathbb{D}^2(p_1, q) & q \geq 2 \end{cases}$$

Proof. For part (1), that the manifolds H_0, H_1, H_2 are genus two handlebodies follows from Lemma 3.7, so the pairs (H_1, J) and (H_2, J) are simple by definition and so (1)(a) holds.

By a similar argument both $H_0 \cup H_1$ and $H_0 \cup H_2$ are handlebodies. Now, if $V_1 \subset H_1$, $V_2 \subset H_2$ are companion solid tori of the power circles $\omega_1' \subset T_1'$, $\omega_2' \subset T_2'$ (see Fig. 11(a)) then, by Lemma 6.3, $H' = V_1 \cup H_0 \cup V_2$ is homeomorphic to H, so by Lemma 3.5(2) the circles ω_1' and ω_2' are basic circles in H_0. Similarly, by Lemma 6.3 ω_1' is primitive in $H_2 \cup H_0$ and ω_2' is primitive in $H_1 \cup H_0$, and if the pair (H_0, J) is simple and $\gamma_i \subset T_i'$ is a power circle in H_0 then $\Delta(\omega_i', \gamma_i) = 1$ by Lemma 6.2(5), and we also have that γ_i is primitive in H_i. Therefore parts (1)(b) and (1)(c) hold.

For part (1)(d), if α_1 is not isotopic to ω_1 then by Lemma 3.6 α_1 is not a power circle in H. If α_1 is a primitive circle in H then by Lemma 6.3 the manifold $H \cup_{\alpha_1} V$ obtained by gluing a solid torus to H along an annular neighborhood of α_1 in T_1, so that α_1 runs $p \geq 2$ times around V, is a genus two handlebody which, as (H_0, J) is a non-trivial pair, contains three mutually disjoint non-parallel and not boundary parallel once-punctured tori with boundary slope J, contradicting Lemma 3.9. Therefore α_1 is also not primitive in H, so by Lemma 3.3(1) the surface $\partial H \setminus \alpha_1$ is incompressible in H, and so by the 2-handle addition theorem the manifold $H(\alpha_1)$ is irreducible with incompressible torus boundary. The remaining part of (1)(d) follows from Lemma 6.7.

For part (2), if (H, J) is a double pair of type (p_1, p_2) then, by construction, there are power p_i circles $\omega_i \subset T_i$ and so the hypothesis of part (1) is satisfied with $(H_0, J) = (T_1' \times I, J)$ a trivial pair; therefore by (1)(a) the torus $T = T_1'$ separates H into simple pairs (H_1, J) and (H_2, J) having the claimed types. Conversely, if a once-punctured torus $T \subset H$ exists that separates H into simple pairs (H_1, J) and (H_2, J) of types $(1, p_1)$ and $(1, p_2)$, respectively, then again T_1 and T_2 contain power p_1 and p_2 circles, so by (1) and (1)(a), along with Lemma 3.9(1), we have that the induced once-punctured tori T_1' and T_2' are parallel in H to T, so (H_0, J) is a trivial pair and so (H, J) is a double pair of type (p_1, p_2). In particular, (2)(a) holds.

Since for any two non-separating circles $\alpha, \beta \subset T$, in $H_0 \approx T \times [-1, 1]$ the circles $\alpha \times \{-1\} \subset T \times \{-1\}$ and $\beta \times \{1\} \subset T \times \{1\}$ are basic circles iff α and β intersect transversely in T in one point, (2)(b) follows from (1)(b) and (1)(c), while (2)(c) follows from the identity $H(\alpha_1) = \mathbb{A}^2(p_2) \cup_{\widehat{T}} H_1(\alpha_1)$. □

6.4 Primitive pairs

Recall that a non-trivial pair (H, J) with $\partial H = T_1 \cup_J T_2$ is *primitive* if there are circles $\alpha_1 \subset T_1$ and $\alpha_2 \subset T_2$ which are primitive and coannular in H. In this section we use primitive pairs to analyze the structure of non-minimal pairs with power or Seifert circles.

Lemma 6.9. *If (H, J) is a primitive pair with $\partial H = T_1 \cup_J T_2$, $A \subset H$ is any annulus with boundary a pair of primitive circles $\alpha_1 \subset T_1$, $\alpha_2 \subset T_2$, and $\beta_i \subset T_i$ is any non-trivial circle with $\Delta(\beta_i, \alpha_i) \geq 1$, then*

1. *the manifold $H(\beta_i)$ is irreducible and boundary irreducible, and if $\Delta(\beta_i, \alpha_i) \geq 2$ then $H(\beta_i)$ is toroidal,*

2. *any incompressible, non-boundary parallel annulus in H with boundary in $\partial H \setminus J$ is isotopic to A, and any circle in T_i which is primitive in H is isotopic to α_i.*

Proof. Assuming (1) holds, if $\beta_i \subset T_i$ is a primitive or power circle in H then $H(\beta_i)$ has compressible boundary by Lemma 3.3(1)(b), hence β_i must be isotopic to α_i; also any incompressible and non-boundary parallel annulus $B \subset H$ with boundary in $\partial H \setminus J$ is either a companion or a non-separating annulus in H, hence each component of ∂B is a primitive circle in H by Lemmas 3.2, 3.3 and the argument above, and so B must be isotopic to A in H by Lemma 3.4(4). Thus part (2) follows from part (1).

Suppose now for definiteness that $\beta_1 \subset T_1$ is any non-trivial circle such that $\Delta(\beta_1, \alpha_1) \geq 1$. By Lemma 3.7(1) the manifold $H(J)$ is irreducible with incompressible boundary $\widehat{T}_1 \sqcup \widehat{T}_2$. Set $H(J)(\alpha_1) = H(J) \cup V_1$, where V_1 is a solid torus attached to $H(J)$ along $\widehat{T}_1$ so that α_1 bounds a disk in V_1. Then $H(J)(\alpha_1) = H(\alpha_1)$ is a solid torus with meridian disk $\widehat{A} \subset H(\alpha_1)$ which intersects the core K_1 of V_1 minimally in one point, so in $H(\alpha_1)$ the knot K_1 has wrapping number one and exterior $H(J) \subset H(\alpha_1)$.

Since the pair (H,J) is not trivial, by Lemma 3.7(4) the knot K_1 is not a core of $H(\alpha_1)$. Therefore K_1 is a locally knotted core of the solid torus $H(\alpha_1)$, that is, the torus $\widehat{F} \subset H(J)$ obtained as the frontier of $M = N(A \cup \partial H(J)) \subset H(J)$ is essential and separates $H(J)$ into two components X and M, where X is the exterior of a non-trivial knot in $\mathbb{S}^3$ (ie, of the local knot tied along the core of $H(\alpha_1)$), and M can be identified with a Seifert fiber space of the form $P \times \mathbb{S}^1$, P a pair of pants, such that $\partial M = \widehat{T}_1 \sqcup \widehat{T}_2 \sqcup \widehat{F}$ and the annulus $A \subset M$ is fibered. Thus $H(\beta_1) = X \cup_{\widehat{F}} M(\beta_1)$ is irreducible and boundary irreducible.

Since $M(\beta_1) \approx \mathbb{A}^2(q)$ for $q = \Delta(\alpha_1, \beta_1) \geq 1$ and $\partial M(\beta_1) = \widehat{T}_2 \sqcup \widehat{F}$, if $q \geq 2$ then $\widehat{T}_2$ and $\widehat{F}$ are not mutually parallel in $M(\beta)$ and so the torus $\widehat{F}$ is essential in $H(\beta)$. Therefore (1) holds. □

In the following result we determine the structure of a general pair (H,J) for which there is a circle $\gamma \subset \partial H \setminus J$ which is either a power circle (eg if (H,J) is a simple, double or maximal pair) or whose complement $\partial H \setminus \gamma$ contains a power circle (eg if γ is a Seifert circle).

Lemma 6.10. *Let (H,J) be a pair with $\partial H = T_1 \cup_J T_2$ and $T \subset H$ any once-punctured torus with $\partial T = J$ which separates H into non-trivial pairs (H_1,J) and (H_2,J) with $\partial H_i = T \cup T_i$.*

1. *If $\omega_1 \subset T_1$ is a power circle in H then either ω_1 is a power circle in H_1 or the pair (H_1,J) is primitive with ω_1 a primitive circle in H_1 and coannular in H_1 to some circle $\omega_1' \subset T$ which is a power circle in H_2.*

2. *If $\alpha_1 \subset T_1$ is a non-separating circle such that the surface $\partial H \setminus \alpha_1 \subset H$ is incompressible and contains a circle β which is a power circle in H, as is*

the case when α_1 is a Seifert circle in H, then either $\beta \subset T_2$ or each pair (H_1, J) and (H_2, J) is a simple or double pair; in particular,

(a) there is a circle in T_2 which is a power circle in H,

(b) if α_1 is a Seifert circle in H then α_1 is a primitive circle in H_1 and there is a circle in T_2 which is a power circle in H_2.

Proof. For part (1), by Lemma 3.3(2) there is a companion annulus A for ω_1 in H. As A and T are incompressible in H, A can be isotoped so as to intersect T minimally, so that $A \cap T$ consists of circles which are non-trivial in A and T. If $A \cap T = \emptyset$ then ω_1 is a power in H_1 by Lemma 3.3(2), so assume that $A \cap T \neq \emptyset$. Then $A \cap H_1$ has an annulus component A_1 with $\partial_1 A_1 = \omega_1$ and $\partial_2 A_1 = \omega_1' \subset T$, and the component A_2 of $A \cap H_2$ with $\partial_1 A_2 = \omega_1'$ is, by minimality of $A \cap T$, a companion annulus for ω_1' in H_2; thus ω_1' is a power circle in H_2 by Lemma 3.3(2). If $T' \subset H_2$ is the once-punctured torus induced by ω_1' then, by Lemma 3.7(2), $H_2|T'$ consists of two handlebodies H_2', H_2'', say with $T' \subset \partial H_2'$, so that the pair (H_2', T') is simple. Similarly, $H|T'$ consists of two handlebodies $H_1 \cup_T H_2'$ and H_2'' and so, by Lemma 6.3 applied to the pairs (H_1, J) and (H_2', J), the circle ω_1', and hence ω_1, are primitive circles in H_1; therefore the pair (H_1, J) is primitive.

For part (2), let $\beta \subset \partial H \setminus \alpha_1$ be a power circle in H. Notice that if α_1 is a Seifert circle in H then by Lemma 6.7 the surface $\partial H \setminus \alpha_1 \subset H$ is incompressible and contains such a power circle β; in particular, by Lemma 3.3(1) α_1 is neither a primitive nor power circle in H.

We assume that β has been isotoped in $\partial H \setminus \alpha_1$ so as to intersect J minimally. As α_1 is not a power circle in H, if $\beta \cap J = \emptyset$ then $\beta \subset T_2$.

Suppose now that $\beta \cap J \neq \emptyset$, and let $B \subset H$ be a companion annulus for β which is disjoint from α_1 and intersects T minimally, so that the graphs of intersection $G_T = B \cap T \subset T$ and $G_B = B \cap T \subset B$ are nonempty. Since $\partial H \setminus \alpha_1$ is incompressible in H, the minimality of $B \cap T$ implies that if $e \subset B \cap T$ is an arc that bounds a trivial disk face D in B (T, resp.) then e is essential in T (B, resp.) and so D is a boundary compression disk for T (B, resp.) in H.

If the graph $G_T = B \cap T \subset T$ has a trivial disk face D_T then boundary compressing B along D_T produces a non-trivial separating disk in H with boundary in $\partial H \setminus \alpha_1$, contradicting the incompressibility of $\partial H \setminus \alpha_1$ in H; therefore the graph G_T has no trivial disk faces.

If the graph $G_B = B \cap T \subset B$ has a trivial disk face D_B and $D_B \subset H_i$ then D_B intersects J minimally in 2 points by Lemma 2.1(3) and so the pair (H_i, J) is simple by Lemma 3.11. If $D_B \subset H_1$ then, as α_1 is disjoint from ∂B, α_1 is disjoint from $D_B \subset B$ and so by Lemma 6.2(5) α_1 is disjoint and hence isotopic in T_1 to the power circle of the simple pair (H_1, J), which is not the case. Therefore $D_B \subset$

H_2 and so the pair (H_2,J) is simple, whence T_2 contains a power circle in H_2 by Lemma 6.2(1).

Otherwise the graphs G_T and G_B are essential, so G_B consists of spanning arcs that cut B into a collection of 4-sided disk faces, alternately lying in H_1 and H_2. By minimality of $B\cap T$ any such disk face of G_B in H_2 intersects J minimally in 4 points; therefore each pair (H_1,J) and (H_2,J) is a simple or double pair by Lemma 6.1, so again T_2 contains a power circle in H_2. Thus (2)(a) holds.

Observe now that $H(\alpha_1)=H_2(J)\cup_{\widehat{T}}H_1(\alpha_1)$, where $H_2(J)$ is an irreducible and boundary irreducible manifold by Lemma 3.7. If α_1 is a Seifert circle in H then $H(\alpha_1)=\mathbb{D}^2(*,*)$ is an irreducible and atoroidal manifold, hence $\widehat{T}$ bounds a solid torus in $H(\alpha_1)$ and so $H_1(\alpha)$ must be a solid torus, so α_1 is primitive in H_1. By (2)(a) there is a circle $\gamma\subset T_2$ which is a power circle in H. If γ is not a power circle in H_2 then by (1) the pair (H_2,J) is primitive, with $\gamma\subset T_2$ a primitive circle in H_2 and coannular to a circle $\gamma'\subset T$ which is a power $q\geq 2$ circle in H_1. By Lemma 3.4 the circles α_1 and γ' are separated in H_1, hence the meridian disk of the solid torus $H_1(\alpha_1)$ intersects γ' minimally in $q\geq 2$ points, which by Lemma 6.9(1) implies that $H(\alpha_1)=H_2(J)\cup_{\widehat{T}}H_1(\alpha_1)=\mathbb{D}^2(*,*)$ is a toroidal manifold, a contradiction. Therefore $\gamma\subset T_2$ is a power circle in H_2 and so (2)(b) holds. □

7 The case $|\mathbb{T}|=6$

In this section we assume that $K\subset S^3$ is a hyperbolic knot and $\mathbb{T}=T_1\sqcup\cdots\sqcup T_N$ a collection of N mutually disjoint and non-parallel once-punctured tori in X_K, initially considering several special cases with $N\leq 5$ before discussing the case $N=6$ in detail.

For the rest of this section we extend each once-punctured torus $T_i\subset X_K$ up to the knot K via annuli in $N(K)$ with disjoint interiors, so that $\partial T_i=K$ and $\operatorname{int}(T_i)\cap\operatorname{int}(T_j)=\emptyset$ for $i\neq j$; for simplicity we will continue to say that the T_i's are mutually disjoint.

Using the notation set up in §6.1, we represent and label the regions $R_{i,i+1}$ as in Fig. 12 (where we take $N=6$), each of which is a handlebody by Lemma 4.3. In particular, if a pair $(R_{i,i+1},K)$ is simple then its core K_i has index $p_i\geq 2$ and its power p_i circles $\omega_i\subset T_i$ and $\omega_i'\subset T_{i+1}$ cobound annuli $A_i,A_i'\subset R_{i,i+1}\setminus\operatorname{int}N(K_i)$ with $\partial_1A_i=\omega_i$, $\partial_1A_i'=\omega_i'$ and circles $\partial_2A_i,\partial_2A_i'\subset\partial N(K_i)$ of slope a_i/p_i relative to $N(K_i)$, where $\gcd(a_i,p_i)=1$, so that $(R_{i,i+1},K)$ is a pair of type $(0,1;a_i,p_i)$.

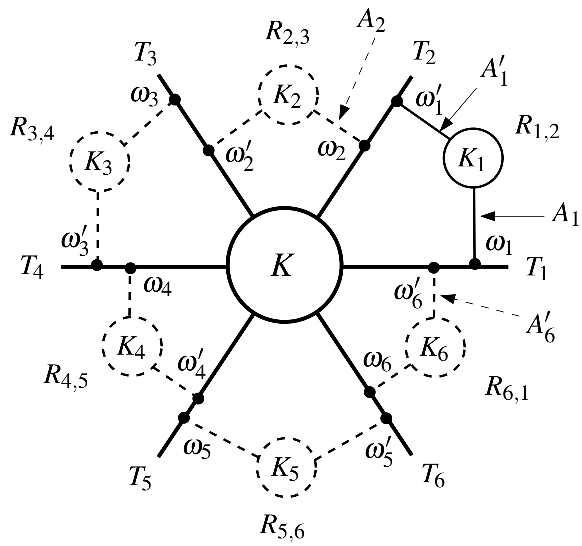

Figure 12: The complementary regions $R_{i,i+1}$ of $\mathbb{T} \subset X_K$.

7.1 Core knots and hyperbolic Eudave-Muñoz knots

The next result establishes a connection between the core knots K_i produced by the collection $\mathbb{T}$ and the family of hyperbolic Eudave-Muñoz knots under some conditions.

Lemma 7.1. *Let $K \subset \mathbb{S}^3$ be a hyperbolic knot that bounds a collection $\mathbb{T} = T_1 \cup T_2 \cup T_3$ of mutually disjoint and non-parallel once-punctured tori, such that $R_{1,2}$ is a handlebody and $(R_{1,2}, K)$ a simple pair with core knot K_1, and the regions $R_{1,3}$ and $R_{3,2}$ are not handlebodies. Let $V_1 = N(K_1) \cup N(A_1) \subset R_{1,2}$ be a solid torus neighborhood of K_1 and identify X_{K_1} with $\mathbb{S}^3 \setminus int V_1$, so that ω_1 is a non-integral slope in ∂X_{K_1} of the form a_1/p_1. Then,*

1. *the twice-punctured torus $F = cl(T_1 \cup T_3 \setminus V_1) \subset X_{K_1}$ is essential in X_{K_1},*
2. *$X_{K_1}(\omega_1)$ is an irreducible manifold and $\widehat{F} \subset X_{K_1}(\omega_1)$ is an incompressible separating torus,*
3. *if $T_a \subset R_{2,3}$ and $T_b \subset R_{3,1}$ are once-punctured tori bounded by K which are not parallel to T_2, T_3 and T_1, T_3, respectively, then*
   (a) K_1 is a hyperbolic Eudave-Muñoz knot of index $p_1 = 2$,

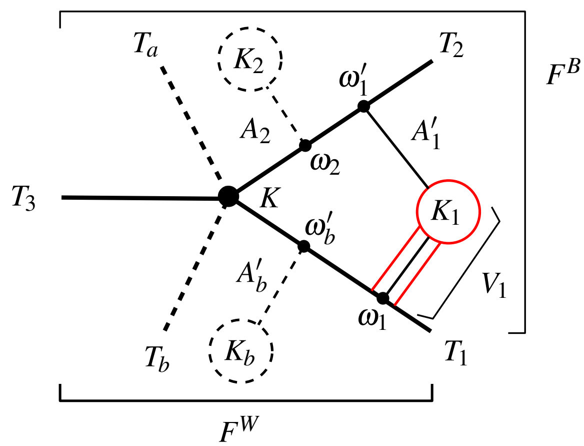

Figure 13: The regions F^B and F^W in $X_{K_1} = \mathbb{S}^3 \setminus \operatorname{int} V_1$.

(b) there are circles $\gamma', \gamma'' \subset T_3$ with $\Delta(\gamma', \gamma'') \neq 0$ which are power circles in $R_{a,3}, R_{3,b}$, respectively; in particular, if any of the pairs $(R_{a,3}, K)$ or $(R_{3,b}, K)$ is minimal then it is simple,

(c) the regions $R_{1,a}$ and $R_{b,2}$ are handlebodies,

(d) if each of the pairs $(R_{2,a}, K)$ and $(R_{b,1}, K)$ is simple of index 2 then the region $R_{b,a}$ is a handlebody.

Proof. Let $F^B, F^W \subset X_{K_1}$ be the closures of the components of $X_{K_1} \setminus F$, with $F^W = R_{3,1}$; the situation is represented in Fig. 13. By Lemma 6.3 F^B is homeomorphic to $R_{2,3}$, while by Lemma 4.1(1) the regions $R_{2,3}$ and $R_{3,1}$ are handlebodies; therefore F^B and F^W are handlebodies.

By Lemmas 5.1 and 6.3, the circle $\omega_1 \subset T_1$ is neither primitive nor a power in $R_{3,1}$. Therefore the surface F is incompressible in $R_{3,1}$ by Lemma 3.3(1) and so, by the 2-handle addition theorem, $F^W(\omega_1) = R_{3,1}(\omega_1)$ is an irreducible manifold with incompressible boundary the torus $\widehat{F}$.

Using the solid torus neighborhood $V_1' = N(K_1) \cup N(A_1') \subset R_{1,2}$ of K_1, it follows in a similar way that ω_1' is neither a primitive nor power circle in $R_{2,3}$, and hence that $\partial R_{2,3} \setminus \omega_1'$ is incompressible in $R_{2,3}$ and $R_{2,3}(\omega_1')$ is an irreducible and boundary irreducible manifold.

Since the homeomorphism between F^B and $R_{2,3}$ identifies F with the surface $\partial R_{2,3} \setminus \operatorname{int} N(\omega_1')$ and the slope ω_1 of the core of the annulus $V_1 \cap F_B$ with ω_1', we have that F is incompressible in X_{K_1} and

$$(*) \quad X_{K_1}(\omega_1) = F^W(\omega_1) \cup_{\widehat{F}} F^B(\omega_1) \approx R_{3,1}(\omega_1) \cup_\partial R_{2,3}(\omega_1')$$

is irreducible with $\widehat{F} \subset X_{K_1}(\omega_1)$ an incompressible separating torus, so (1) and (2) hold. In particular, K_1 is not a torus knot, so by [17] K_1 is either a satellite or hyperbolic knot.

For part (3) observe that, by Lemma 3.7(2), $(R_{2,a}, K)$, $(R_{a,3}, K)$, $(R_{3,b}, K)$ and $(R_{b,1}, K)$ are all non-trivial pairs. As the boundary slope $\omega_1 \subset \partial X_{K_1}$ of F is non-integral, if K_1 is a satellite knot then by Lemma 5.2 there is a circle $\gamma \subset F$, not parallel to ∂F, which is a power circle in F^B and F^W. Via the homeomorphism $F^B \approx R_{2,3}$, γ corresponds to a circle in $\partial R_{2,3} \setminus \omega_1'$ which is a power circle in $R_{2,3}$, so by Lemma 6.10(2)(a) there is a circle $\gamma' \subset T_3$ which is a power circle in $F^B \approx R_{2,3}$. A similar argument applied to $F^W = R_{3,1}$ shows that there is a circle $\gamma'' \subset T_3$ which is a power circle in F^W. However, by Lemma 3.6, the circles γ and γ' are isotopic in $\partial F^B(\omega_1')$, while γ and γ'' are isotopic in $\partial F^W(\omega_1)$. But then γ' and γ'' must be isotopic in T_3, which by Lemma 5.1 cannot be the case since K is a hyperbolic knot.

Therefore K_1 must be a hyperbolic knot, so by [9, Theorem 1.1] K_1 is a hyperbolic Eudave-Muñoz knot and the slope a_1/p_1 of ∂F_1 is half-integral, whence $p_1 = 2$. By [5, Theorem 2.1] and the invariance argument used in [5, Proposition 2.2], the closed torus $\widehat{F} \subset K_1(\omega_1)$ is unique up to isotopy and separates $X_{K_1}(\omega_1)$ into two Seifert fiber spaces $F^B(\omega_1)$ and $F^W(\omega_1)$ of type $\mathbb{D}^2(*,*)$ (the uniqueness of the torus $\widehat{F} \subset K_1(\omega_1)$ also follows from the fact [9] that the regular fibers of the two Seifert fiber spaces $\mathbb{D}^2(*,*)$ in $K_1(\omega_1)$ intersect transversely in one point). Therefore, by Lemma 6.10(2)(b), there are circles $\gamma', \gamma'' \subset T_3$ which are power circles in $R_{a,3}, R_{3,b}$, respectively, where $\Delta(\gamma', \gamma'') \neq 0$ by Lemma 5.1. And whichever pair $(R_{a,3}, K)$ or $(R_{3,b}, K)$ is minimal, by Lemma 6.2 it must be simple.

Moreover, as ω_1 is a Seifert circle in $F^W = R_{3,1}$, by Lemma 6.10(2)(b) the circle ω_1 is primitive in $R_{b,1}$ and so $R_{b,2}$ is a handlebody by Lemma 6.3. Since $F^B(\omega_1)$ corresponds to $R_{2,3}(\omega_1')$, in a similar way it follows that ω_1' is primitive in $R_{2,a}$ and $R_{1,a}$ is a handlebody. Therefore (3)(a), 3(b), and 3(c) hold.

For (3)(d), if each of the pairs $(R_{b,1}, K)$ and $(R_{2,a}, K)$ is simple of index 2 then each circle $\partial_2 A_2 \subset \partial N(K_2)$ and $\partial_2 A_b' \subset \partial N(K_b)$ (see Fig. 13) bounds a Moebius band $B_2 \subset N(K_2)$ and $B_b \subset N(K_b)$. By (3)(c) and Lemma 3.5(1) the circles $\omega_1 = \partial_1 A_b'$ and $\omega_1' = \partial_1 A_2$ are primitive in $R_{1,2}$ and so by Lemma 6.4(3) there is a slope s_1 in $\partial N(K_1)$ which along with $\omega_b' \sqcup \omega_2$ cobounds a pair of pants P_1 in $R_{1,2} \setminus \operatorname{int} N(K_1)$. Thus the slope $s_1 \subset \partial N(K_1)$ bounds the once-punctured Klein bottle $B_2 \cup P_1 \cup B_b$ in the exterior $\mathbb{S}^3 \setminus \operatorname{int} N(K_1)$ of the hyperbolic knot K_1 and so by [8, Theorem 1.3] the slope s_1 is integral; therefore the circles ω_b', ω_2 are basic in $R_{1,2}$ by Lemma 6.4(4) and hence $R_{b,a}$ is a handlebody by Lemma 3.5(2). □

7.2 Heegaard splittings of $\mathbb{S}^3$

For the rest of Section 7 we consider the case $N = 6$ exclusively. In this section we prove that some pair of complementary regions $R_{i,i+3}, R_{i+3,i}$ form a Heegaard splitting of $\mathbb{S}^3$. For convenience we summarize below a number of properties of the regions $R_{i,j}$.

Lemma 7.2. *1. Each region $R_{i,i+1}$ is a handlebody and each pair $(R_{i,i+1}, K)$ is minimal and non-trivial.*

2. *For some $i \in \{1,2\}$ each pair $(R_{i,i+1}, K)$, $(R_{i+2,i+3}, K)$ and $(R_{i+4,i+5}, K)$ is simple.*

3. *Each region $R_{i,i+2}$ is a handlebody.*

4. *No pair $(R_{i,i+1}, K)$ is a double pair, and if a region $R_{i,i+3}$ is a handlebody then $(R_{i,i+3}, K)$ is neither a simple nor a double pair and $(R_{i+1,i+2}, K)$ is a basic non-primitive pair.*

5. *The region $R_{i,i+3}$ is a handlebody iff the pair $(R_{i,i+1}, K)$ is simple and the power circle $\omega_i' \subset T_{i+1}$ of $R_{i,i+1}$ is primitive in $R_{i+1,i+3}$, and if $R_{i,i+3}$ is a handlebody then $(R_{i+2,i+3}, K)$ is also a simple pair. Thus, if all regions $R_{i,i+3}$ are handlebodies then all pairs $(R_{i,i+1}, K)$ are basic and simple, and if $(R_{i,i+1}, K)$ is not a simple pair then the regions $R_{i,i+3}$ and $R_{i-2,i}$ are not handlebodies.*

Proof. Part (1) follows directly from Lemma 4.3 since the degree of each vertex of G_Q is 6. Also, by Lemmas 2.3(1) and 4.2 there is a vertex v in $\overline{G}_Q$ of degree 3 around which there are 3 incident bigon disk faces of G_Q located in alternating regions; thus (2) holds. We also have that for each i the region $R_{i,i+2}$ contains no bigon disk faces of $G_Q^{i,i+2}$, so the graph $G_Q^{i,i+2,i+4}$ is reduced with each vertex of degree 3 and so by Lemma 4.1(3) the region $R_{i,i+2} \subset X_K$ is a handlebody; therefore (3) holds.

If the region $R_{i,i+3}$ is a handlebody then by Corollary 3.10 the pairs $(R_{i,i+1}, K)$ and $(R_{i+2,i+3}, K)$ are simple, and by Lemma 6.8(1)(b) the power circles $\omega_i' \subset T_{i+1}$ and $\omega_{i+2} \subset T_{i+2}$ are basic in $R_{i+1,i+2}$. Therefore, in $R_{i+1,i+2}$, the circles ω_i', ω_{i+2} are primitive but not homotopic to each other, hence not coannular, which by Lemma 6.9(2) implies that the pair $(R_{i+1,i+2}, K)$ is not primitive. The remaining parts of (4) and (5) follow from (1) and Lemmas 3.7(3) and 6.8. □

Lemma 7.3. *At most one pair $(R_{i,i+1}, J)$ may not be simple.*

Proof. Suppose, for definiteness, that the pair $(R_{1,2}, K)$ is not simple. Then the region $R_{1,4}$ is not a handlebody by Lemma 7.2(5), so $R_{4,1}$ is a handlebody by

Lemma 4.1(1). By Lemma 7.2(4), neither $(R_{1,2}, K)$ nor $(R_{4,1}, K)$ is a simple or double pair; as the pair $(R_{2,4}, K)$ is not minimal, by Lemma 3.9 it is not simple. Therefore by Lemma 4.2 the graph $G_Q^{1,2,4}$ has no bigon disk faces, and by Lemma 6.1 it has no 4-sided disk faces in $R_{1,2}$ or $R_{4,1}$. It follows that $G_Q^{1,2,4}$ is a reduced planar graph with each vertex of degree 3, which by Lemma 2.3 must have 4-sided disk faces, all of which must lie in the region $R_{2,4}$. Therefore $(R_{2,4}, K)$ is a double pair by Lemma 6.1 and so the pair $(R_{3,4}, K)$ is simple by Lemma 6.8(2).

A similar argument applied to the graph $G_Q^{1,2,5}$ shows that the pair $(R_{5,6}, K)$ is also simple. Since by Lemma 7.2(2) the pairs $(R_{2,3}, K), (R_{4,5}, K)$ and $(R_{6,1}, K)$ must be simple, the lemma follows. □

Lemma 7.4. *If the region $R_{1,4}$ is not a handlebody then*

1. *all the pairs $(R_{i,i+1}, K)$ are simple,*
2. *the core knots $K_1 \subset R_{1,2}$ and $K_3 \subset R_{3,4}$ are hyperbolic Eudave-Muñoz knots of indices $p_1 = 2 = p_3$,*
3. *all regions $R_{i,i+3} \neq R_{1,4}$ are handlebodies.*

Proof. Recall that if the region $R_{i,i+3}$ is a handlebody then by Lemma 7.2(5) the pairs $(R_{i,i+1}, K)$ and $(R_{i+2,i+3}, K)$ are simple.

Since $R_{1,4}$ is not a handlebody, by Lemma 4.1(1) the region $R_{4,1}$ is a handlebody, hence the pairs $(R_{4,5}, K)$ and $(R_{6,1}, K)$ are simple. By Lemma 7.3 we may assume that one of the pairs $(R_{1,2}, K)$ or $(R_{3,4}, K)$, say $(R_{1,2}, K)$, is simple. Thus at most one of the remaining pairs $(R_{2,3}, K)$, $(R_{3,4}, K)$, or $(R_{5,6}, K)$ may not be simple.

Now, by Lemma 7.2(3) the region $R_{2,4}$ is a handlebody, while by Lemma 3.9 $R_{4,2}$ is not a handlebody. Therefore, by Lemma 7.1(3) applied to the simple pair $(R_{1,2}, K)$ and the collection of tori T_1, T_2, T_4 with $T_a = T_3$ and $T_b = T_5$, the knot K_1 is a hyperbolic Eudave-Muñoz knot of index $p_1 = 2$, the minimal pair $(R_{3,4}, K)$ is simple, so the core knot K_3 is defined, and $R_{5,2}$ is a handlebody and so the pair $(R_{5,6}, K)$ is simple. By symmetry, K_3 is also a hyperbolic Eudave-Muñoz knot of index $p_3 = 2$ and $R_{3,6}$ is a handlebody.

If $R_{2,5}$ is not a handlebody then applying the argument above applied to the simple pair $(R_{4,5}, K)$ shows that $(R_{2,3}, K)$ is a simple pair and the core knots $K_2 \subset R_{2,3}$ and $K_4 \subset R_{4,5}$ are hyperbolic Eudave-Muñoz knots of indices $p_2 = 2 = p_4$, contradicting Lemma 7.1(3)(d) applied to the simple pair $(R_{3,4}, K)$ and the tori T_1, T_3, T_4 and $T_a = T_2, T_b = T_5$. Therefore $R_{2,5}$ is a handlebody, so $(R_{2,3}, K)$ is a simple pair, and in a similar way $R_{6,3}$ is also a handlebody. □

We now combine the results above to obtain a genus two Heegaard splitting of $\mathbb{S}^3$.

Proposition 7.5. *All the pairs $(R_{i,i+1}, K)$ are simple and, without loss of generality, we may assume that all the regions $R_{1,4}, R_{4,1}, R_{5,2}$ and $R_{3,6}$ are handlebodies. In particular, $R_{1,4} \cup_\partial R_{4,1}$ is a genus two Heegaard splitting of $\mathbb{S}^3$ and $\omega_1 \subset T_1$ and $\omega_3' \subset T_4$ are Seifert circles in $R_{4,1}$.*

Proof. That all pairs $(R_{i,i+1}, K)$ are simple follows from Lemma 7.2(5) if all the regions $R_{i,i+3}$ are handlebodies, and otherwise from Lemma 7.4, which also implies that at most one region $R_{i,i+3}$ is not a handlebody, so we may assume that $R_{1,4}, R_{4,1}, R_{5,2}$ and $R_{3,6}$ are handlebodies.

Since $R_{3,6}$ is a handlebody, by Lemma 7.2(5) the circle $\omega_3' \subset T_4$ is primitive in $R_{4,6}$ and hence a Seifert circle in $R_{4,1}$ by Lemma 6.8(1)(d), disjoint from the power circle $\omega_6' \subset T_1 \subset \partial R_{1,4}$. In a similar way, $\omega_1 \subset T_1$ is a Seifert circle in $R_{4,1}$ since $R_{5,2}$ is a handlebody. □

7.3 Heegaard diagrams

In this section we construct the Heegaard diagrams of the genus two Heegaard splittings $R_{1,4} \cup_\partial R_{4,1}$ of $\mathbb{S}^3$ provided in Proposition 7.5. To this end we first obtain specific homeomorphic representations of basic simple pairs and more general pairs with the help of the following result.

Lemma 7.6. *Let S be a closed genus two surface and $a_1, b_1, a_2, b_2, a_0, b_0, c_0 \subset S$ non-trivial circles which intersect minimally as shown in Fig. 14(a), where c_0 separates S into two once-punctured tori S_1, S_2 with $a_i \cup b_i \subset S_i$. Then*

1. *any non-trivial separating circle $c_a \subset S$ which is disjoint from $a_1 \sqcup a_2$ and intersects a_0 minimally in 2 points is obtained by Dehn twisting c_0 along b_0, that is by connecting the endpoints of $2n$ non-trivial arcs in $S_1 \setminus a_1$ and $2n$ non-trivial arcs in $S_2 \setminus a_2$ in one of the two ways shown in Fig. 14(b);*

2. *any non-trivial separating circle $c_b \subset S$ which is disjoint from $b_1 \sqcup b_2$ and intersects b_0 minimally in 2 points is obtained by Dehn twisting c_0 along a_0, that is by connecting the endpoints of $2n$ non-trivial arcs in $S_1 \setminus b_1$ and $2n$ non-trivial arcs in $S_2 \setminus b_2$ in one of the two ways shown in Fig. 14(c).*

Proof. Let $A_0 \subset S$ be a thin annular neighborhood of c_0; we will refer to the components of $S \setminus \operatorname{int} A_0$ as S_1 and S_2, correspondingly, so that $\partial S_1 \sqcup \partial S_2 = \partial A_0$.

Suppose $c_a \subset S$ is a non-trivial separating circle disjoint from $a_1 \sqcup a_2$. Then c_a may be isotoped so as to intersect c_0 minimally, hence to intersect $A_0 \subset S$ minimally into a collection of parallel spanning arcs. The arcs $c_a \cap S_i$, being disjoint from the circle $a_i \subset S_i$, form a disjoint family of mutually parallel non-trivial arcs in S_i. Since $|a_0 \cap a_i| = 1$ it is possible to isotope c_a, if necessary, so that the arcs $c_a \cap S_i$

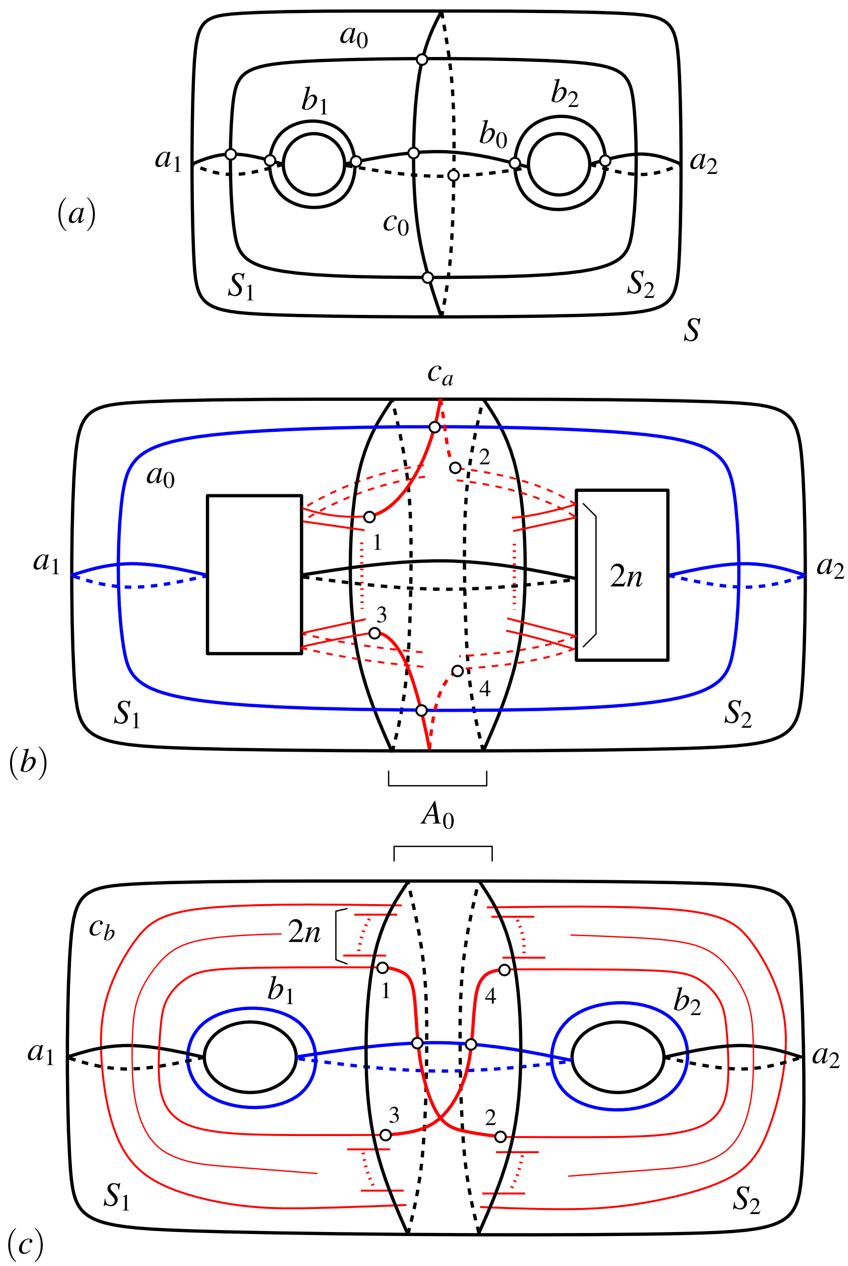

Figure 14: The separating circles $c_a, c_b \subset S$.

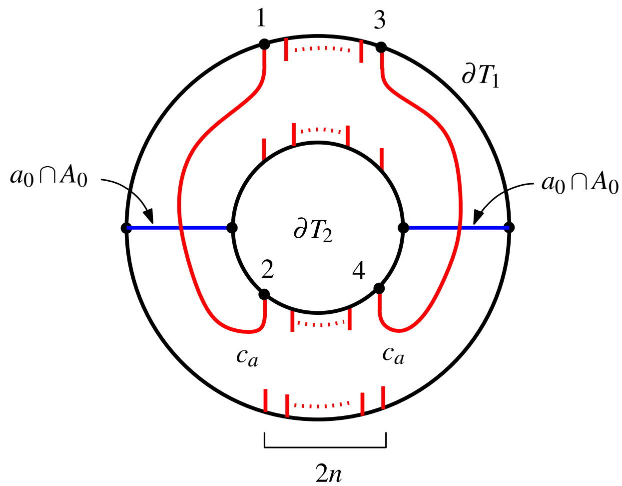

Figure 15: Generators of the arcs $c_a \cap A_0$ in the annulus A_0.

are disjoint from the arc $a_0 \cap S_i$, that is, so that the points $c_a \cap a_0$ lie in the annulus A_0. Also, as c_a separates S, we must have $|c_a \cap S_i| = 2n$ for some integer $n \geq 1$. The situation so far is represented in Fig. 14(b).

In the annulus A_0 the endpoints of the spanning arcs $c_a \cap A_0 \subset A_0$ are distributed around $\partial A_0 = \partial S_1 \sqcup \partial S_2$ and separated by the two arcs $a_0 \cap A_c$ as shown in Fig. 15. Now, the collection of arcs $c_a \cap A_0$ is uniquely determined by one spanning arc connecting a point of $c_a \cap S_1$ with a point of $c_a \cap S_2$. It is not hard to see that the only collections $c_a \cap A_0$ which intersect $a \cap A_c$ minimally in two points are the ones generated from the arc connecting the points 1 and 2 or the arc connecting the points 3 and 4 indicated in Fig. 15, each of which in fact produces a separating circle c_a in S as shown in Fig. 14(b). Therefore part (1) holds, and (2) follows in a similar way. □

We now construct a diagram for the Heegaard splitting $R_{1,4} \cup_\partial R_{4,1}$ as follows. Since $R_{1,4} = R_{1,2} \cup_{T_2} R_{2,3} \cup_{T_3} R_{3,4}$ is a handlebody, by Lemma 6.8(1)(b) the circles $\omega_1' \subset T_2$ and $\omega_3 \subset T_3$ are basic circles in $R_{2,3}$; therefore, as the pair $(R_{2,3}, K)$ is simple, by Lemma 6.5(3), there are unique disks $D, D', D'' \subset R_{2,3}$ such that the 7-tuple $(R_{2,3}, D, D', D'', \omega_1', \omega_3, K)$ is homeomorphic to the 7-tuple $(H, D, D', D'', \alpha_1, \alpha_2, J)$ in Fig. 9(b) (where $p_2 = p = 2$ is used for simplicity).

Since $|\omega_1' \cap D''| = 1$, by Lemma 3.4 $E_{2,3} = \mathrm{fr}\, N(\omega_1' \cup D'') \subset R_{2,3}$ is the unique disk that separates the primitive circles ω_1', ω_3; moreover $E_{2,3}$ intersects D minimally in one arc and the separating circle $K \subset \partial R_{2,3}$ minimally in $4p_2$ points (see

Fig. 9(b), with $J = K$).

Therefore, by Lemma 7.6(2) with $a_1 = \partial D''$, $a_2 = \partial D'$, $b_0 = \partial D$, $b_1 = \omega_1'$, $b_2 = \omega_3$, $c_0 = \partial E_{2,3}$ and $c_b = K$, the 7-tuple $(R_{2,3}, D, D', D'', \omega_1', \omega_3, K)$ is homeomorphic to the 7-tuple shown in Fig. 16(a), where there are two choices for the circle K, while the 6-tuple $(\partial R_{2,3}, \partial D, \omega_1', \omega_3, \partial E_{2,3}, K)$ is homeomorphic to the 6-tuple in Fig. 16(b) by Lemma 7.6(1), where there are two choices for the circle $\partial E_{2,3}$.

Remark 7.7. *(1) If (H,J) is any basic pair and $\alpha_1 \subset T_1$ and $\alpha_2 \subset T_2$ are basic circles in H then, by the 2-handle addition theorem and Lemma 3.4, α_1 and α_2 are separated in H and so the compression disk of $\partial H \setminus \alpha_i$ intersects α_j minimally in one point. It is not hard to see by the argument above that the pair (H,J) must therefore be homeomorphic to the pair $(R_{2,3}, K)$ in Fig. 16(a) obtained by any valid connecting pattern between the endpoints of the arcs $K \cap S_1$ and $K \cap S_2$, and that (H,J) is simple iff it is constructed using the specific connecting schemes in Fig. 16(a).*
(2) By Corollary 3.10 and Lemmas 3.5 and 6.3, any maximal pair (H,J) is homeomorphic to a manifold obtained by attaching solid tori V_1, V_2 along annular neighborhoods of basic circles $\alpha_1 \subset T_1'$, $\alpha_2 \subset T_2'$, respectively, of a nontrivial basic pair (H_0, J) with $\partial H_0 = T_1' \cup_J T_2'$, in such a way that each circle α_i runs at least twice around V_i.

By Lemmas 3.5(2) and 6.3, attaching the companion solid tori $V_1' \subset R_{1,2}$ and $V_3 \subset R_{3,4}$ to $R_{2,3}$ along the circles ω_1' and ω_3, respectively, yields a handlebody homeomorphic to $R_{1,4}$ such that the 5-tuple $(\partial R_{1,4}, \omega_1, \omega_3', \partial E_{2,3}, K)$ is homeomorphic to the 5-tuple $(\partial R_{2,3}, \omega_1', \omega_3, \partial E_{2,3}, K)$ in Fig. 16(b).

Notice that $E_{2,3} \subset R_{2,3}$ becomes a waist disk in $R_{1,4}$ which cuts $R_{1,4}$ into two solid tori $V_1, V_3 \subset R_{1,4}$, and such that $\partial E_{2,3}$ cuts $\partial R_{1,4}$ into two once-punctured tori $S_1 \subset \partial V_1$ and $S_4 \subset \partial V_3$, with $\omega_1 \subset S_1$, $\omega_3' \subset S_4$, and meridian disks $D_1 \subset V_1$ and $D_3 \subset V_3$ with $\partial D_1 \subset S_1$ and $\partial D_3 \subset S_4$. Thus D_1 and D_3 are the compression disks in $R_{1,4}$ of $\partial R_{1,4} \setminus \omega_3'$ and $\partial R_{1,4} \setminus \omega_1$, respectively, which are unique by Lemma 3.3(1)(b). Since in $\partial R_{1,4}$ the circles ω_1, ω_3' are disjoint from $K \cup \partial E_{2,3}$ while ω_4, ω_6' are disjoint from K with $|\omega_1 \cap \omega_6'| = 1 = |\omega_4 \cap \omega_3'|$, it follows that the 7-tuple $(\partial R_{1,4}, \omega_1, \omega_3', \omega_4, \omega_6', \partial E_{2,3}, K)$ is homeomorphic to the one shown in Fig. 16(c).

Let $\partial E_{2,3}^{(1)}$ and $\partial E_{2,3}^{(2)}$ be the versions of the circle $\partial E_{2,3}$ shown in Fig. 16(c) obtained by connecting the endpoints 1 and 2 or 3 and 4 in Fig. 16(b), respectively. It is not hard to see that the automorphism of $\partial R_{1,4}$ obtained by reflecting the surface $\partial R_{1,4}$ across the plane that contains the circles $\omega_6' \sqcup \omega_4$ maps $\partial E_{2,3}^{(i)}$ onto $\partial E_{2,3}^{(j)}$ for $\{i, j\} = \{1, 2\}$.

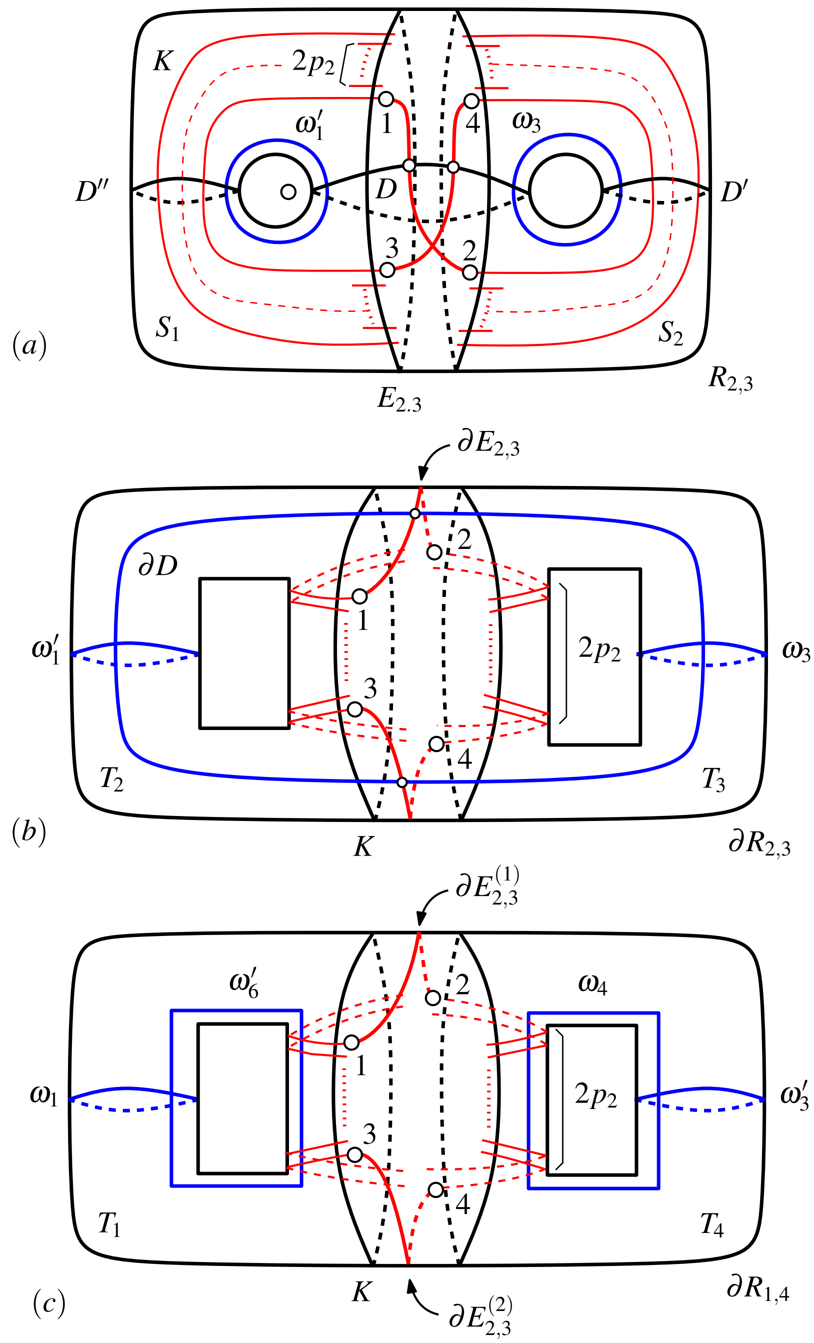

Figure 16: The circles K and $\partial E_{2,3}^{(i)}$ in $\partial R_{2,3}$ and $\partial R_{1,4}$.

Therefore in the sequel we will assume for definiteness that $\partial E_{2,3} = \partial E_{2,3}^{(1)}$, as shown in Fig. 17(a) (where $p_2 = 2$).

In order to obtain the first half of the Heegaard diagram for $R_{1,4} \cup R_{4,1}$, it remains to identify the circles $\partial D_1 \subset S_1$ and $\partial D_3 \subset S_4$ in the version of $\partial R_{1,4} = S_1 \cup_\partial S_4$ shown in Fig. 17(a), where $p_2 = 2$ is used for simplicity. We do this with the help of a specific homological frame for S_1 and S_4.

The oriented circles a_1, b_1 indicated in Fig. 17(b) lie in T_1 and have the minimal intersections $|a_1 \cap b_1| = |a_1 \cap \omega_1| = |b_1 \cap \omega_6'| = 1$ and $|b_1 \cap \omega_1| = 0$. Since a_1, b_1 are disjoint from $\partial E_{2,3}^{(1)}$ and $|\omega_1 \cap \partial D_1| = p_1$, homologically in S_1 we can write $\partial D_1 = p_1 a_1 + q_1 b_1$ for some integer q_1 with $\gcd(p_1, q_1) = 1$.

This and future arrangements can be described as follows: An oriented circle with a box $\boxed{k}$ on top represents a collection of $|k|$ mutually disjoint, parallel circles, oriented in the direction given by the arrows on the circle if $k > 0$, and in the opposite direction if $k < 0$; thus ∂D_1 is the circle obtained as the homological sum of the circle collections with boxes p_1 and q_1 in Fig. 17(b). The circle ∂D_3 is constructed in a similar way as the homological sum of the collection of circles with boxes p_3, q_3, $\gcd(p_3, q_3) = 1$, shown in Fig. 17(c).

The second half of the Heegaard diagram for $R_{1,4} \cup R_{4,1}$ is obtained similarly: A waist disk $E_{5,6} \subset R_{4,1}$ is constructed that separates $R_{4,1}$ into solid tori that contain the power circles ω_4 and ω_6' and have meridian disks D_4 and D_6 with minimal intersections $|D_4 \cap \omega_4| = p_4$, $|D_6 \cap \omega_6'| = p_6$, and $|D_4 \cap \omega_6'| = 0 = |D_6 \cap \omega_4|$. We then use the method of Lemma 7.6(2) (see Fig. 14(c)) to represent the circle $\partial E_{5,6} \subset \partial R_{4,1}$ on top of the diagrams for $\partial R_{1,4} = \partial R_{4,1}$ of Fig. 17.

- We will call the diagram for $\partial R_{4,1}$ obtained by constructing the circle $\partial E_{5,6} = \partial E_{5,6}^{(1)}$ using the endpoints labeled 1 and 2 in Fig. 14(c) a *type 1 diagram*, and a *type 2 diagram* if $\partial E_{5,6} = \partial E_{5,6}^{(2)}$ is constructed using the endpoints labeled 3 and 4 in Fig. 14(c).

The Heegaard diagrams are now uniquely determined up to some number $n \in \mathbb{Z}$ of Dehn twists along the annulus $A_K \subset \partial R_{1,4}$, which we consider in more detail in the sequel. In the meantime, for $n = 0$, Fig. 18(a) shows the circle $\partial E_{5,6}^{(1)}$ of a type 1 diagram for $R_{4,1}$ with $p_5 = 2$, and the circles ∂D_4 and ∂D_6 appear in Fig. 18(b) and (c), as obtained from the construction above.

We summarize our findings in this section in the following result:

Lemma 7.8. *If $K \subset \mathbb{S}^3$ is a genus one hyperbolic knot whose exterior X_K contains 6 mutually disjoint and non-parallel once-punctured tori, then $\mathbb{S}^3$ admits a genus two Heegaard splitting $R_{1,4} \cup_\partial R_{4,1}$ of type 1 or 2 with $K \subset \partial R_{1,4} = \partial R_{4,1}$ a separating circle.* $\square$

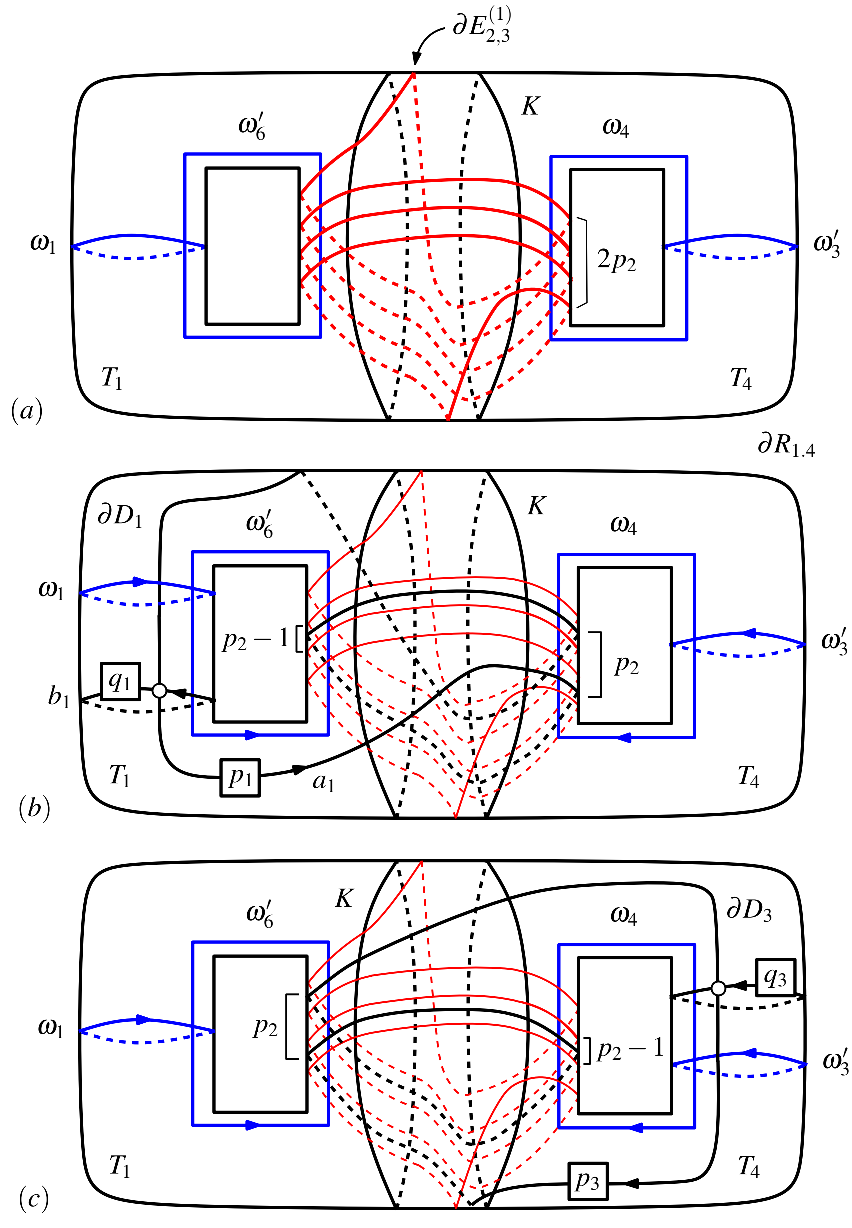

Figure 17: The circles (a) $\partial E_{2,3} = \partial E_{2,3}^{(1)}$ ($p_2 = 2$) and (b,c) $\partial D_1, \partial D_3$ in $\partial R_{1,4}$.

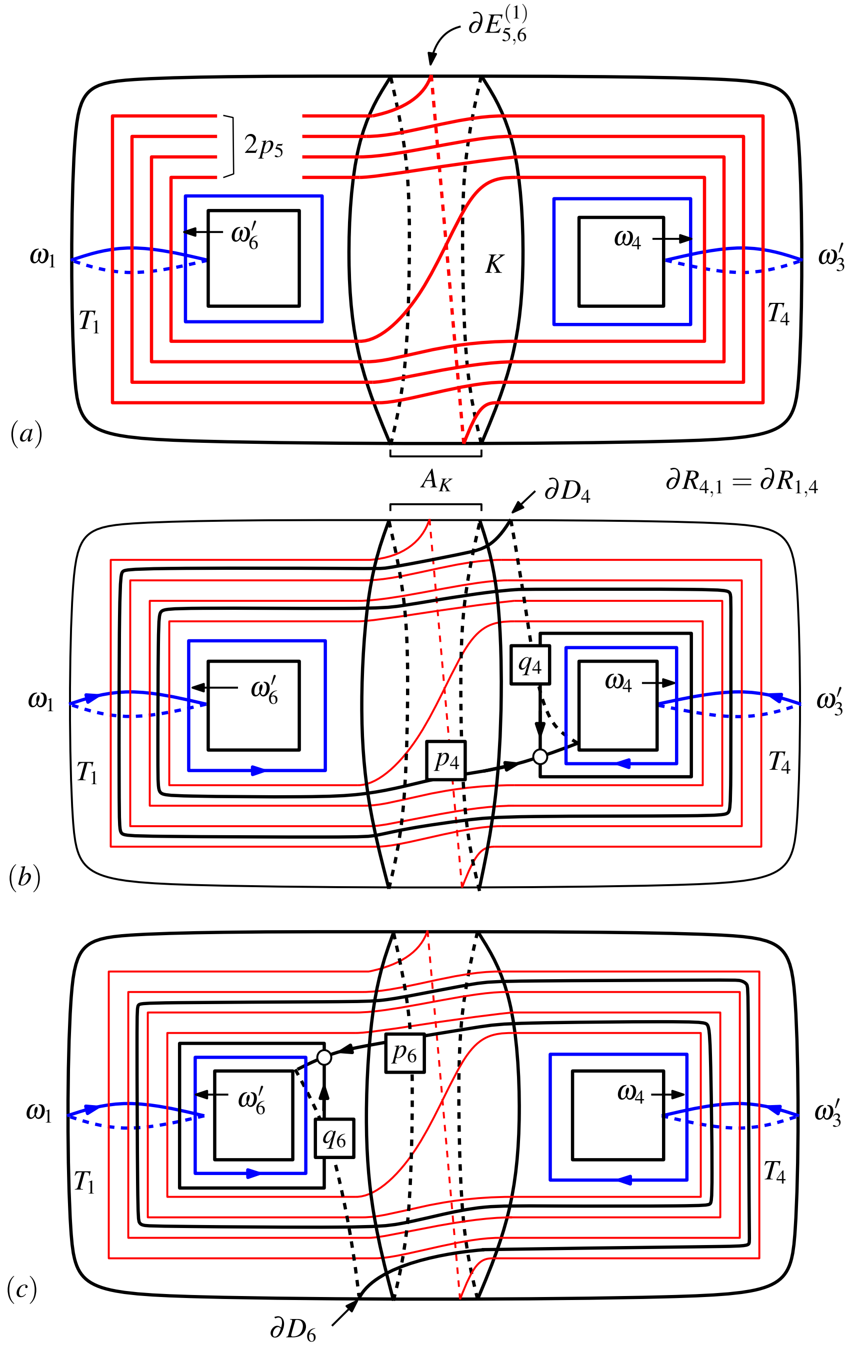

Figure 18: The circles $\partial E^{(1)}_{5,6}$ ($p_5 = 2$) and $\partial D_4, \partial D_6$ in $\partial R_{4,1} = \partial R_{1,4}$ for $n = 0$.

7.4 The type 1 Heegaard diagrams for $R_{1,4} \cup_\partial R_{4,1}$

The identification of $\partial R_{1,4}$ and $\partial R_{4,1}$ is completely determined by the images of the circle pairs $\omega_1 \sqcup \omega_6'$ and $\omega_3' \sqcup \omega_4$ up to some number $n \in \mathbb{Z}$ of Dehn twists along the annular neighborhood $A_K \subset \partial R_{1,4}$ of K shown in Fig. 19(a); the Dehn twists are applied only to the arcs $A_K \cap (\partial D_4 \sqcup \partial D_6)$, where $n > 0$ is taken as the direction indicated by the arrows along the arcs γ, δ in A_K shown in Fig. 19(a).

Fig. 19(a) shows the embeddings of the circles ∂D_1 and ∂D_3 in $\partial R_{1,4}$ obtained with $p_2 = 2$, and the embeddings of ∂D_4 and ∂D_6 are shown in Fig. 19(b) and (c), respectively, with $n = 0$ and $p_5 = 3$.

7.4.1 Fundamental group presentations I

In order to analyze the fundamental group of the manifold $R_{1,4} \cup_\partial R_{4,1}$ and properties of the words represented by circles in the Heegaard surface $\partial R_{1,4}$, we consider here the situation in more general terms.

Let H be a genus two handlebody; its fundamental group is isomorphic to the rank 2 free group $\mathbb{F}_2$. For $i = 1, 2$, let $\gamma_i \subset \partial H$ be disjoint separated power p_i circles with $p_1 \geq 1$ and $p_2 \geq 2$, where if $p_1 = 1$ then γ_1 is taken to be a primitive circle. Thus there is a waist disk D that cuts H into two solid tori V_1, V_2 with $\gamma_i \subset \partial V_i \setminus D$, and by Lemma 3.3(1)(b) the meridian disks $D_1 \subset V_1 \setminus D$ and $D_2 \subset V_2 \setminus D$ are the unique compression disks of $\partial H \setminus \gamma_2$ and $\partial H \setminus \gamma_1$, respectively. Let x_i be a core circle of V_i dual to D_i, so that $\pi_1(H, q) = \langle x_1, x_2 \mid - \rangle$ ($q \in D$).

By Lemma 3.3(2) the companion annulus $A_2 \subset H$ of γ_2 is unique and can be isotoped away from D and into V_2, hence D lies in the handlebody $H_{A_2} \subset H|A_2$ as a waist disk; since by Lemma 3.5(1) the core circle $t_2 \subset \partial H_{A_2}$ of A_2 is primitive in H_{A_2}, we have that $\pi_1(H_{A_2}, q) = \langle x_1, t_2 \mid - \rangle$ ($q \in D$).

The next result now follows from Van Kampen's theorem.

Lemma 7.9. *The map $\pi_1(H_{A_2}, q) \to \pi_1(H, q)$ ($q \in D$) induced by the inclusion $H_{A_2} \subset H$ is an injection given by $x_1 \mapsto x_1$ and $t_2 \mapsto x_2^{p_2}$. In particular, if a circle $\gamma \subset \partial H \setminus \gamma_2$ is represented by the words $w(x_1, t_2) \in \pi_1(H_{A_2}, q) = \langle x_1, t_2 \mid - \rangle$ and $W(x_1, x_2) \in \pi_1(H, q) = \langle x_1, x_2 \mid - \rangle$ ($q \in \gamma \cap D$) then $W(x_1, x_2) = w(x_1, x_2^{p_2})$.* □

Determining which words in the free group $\mathbb{F}_2$ of rank two are primitive will be useful in the sequel. The next result from [3] gives a simple condition satisfied by such words.

Lemma 7.10. *([3]) In any cyclically reduced primitive word in $\mathbb{F}_2 = \langle x_1, x_2 \mid - \rangle$ different from $x_1^{\pm 1}$ or $x_2^{\pm 1}$, for some $\{i, j\} = \{1, 2\}$, the exponents in x_i are all equal to 1 or all equal to -1, while the exponents in x_j are all nonzero of the form m or $m+1$ for some $m \in \mathbb{Z}$.* □

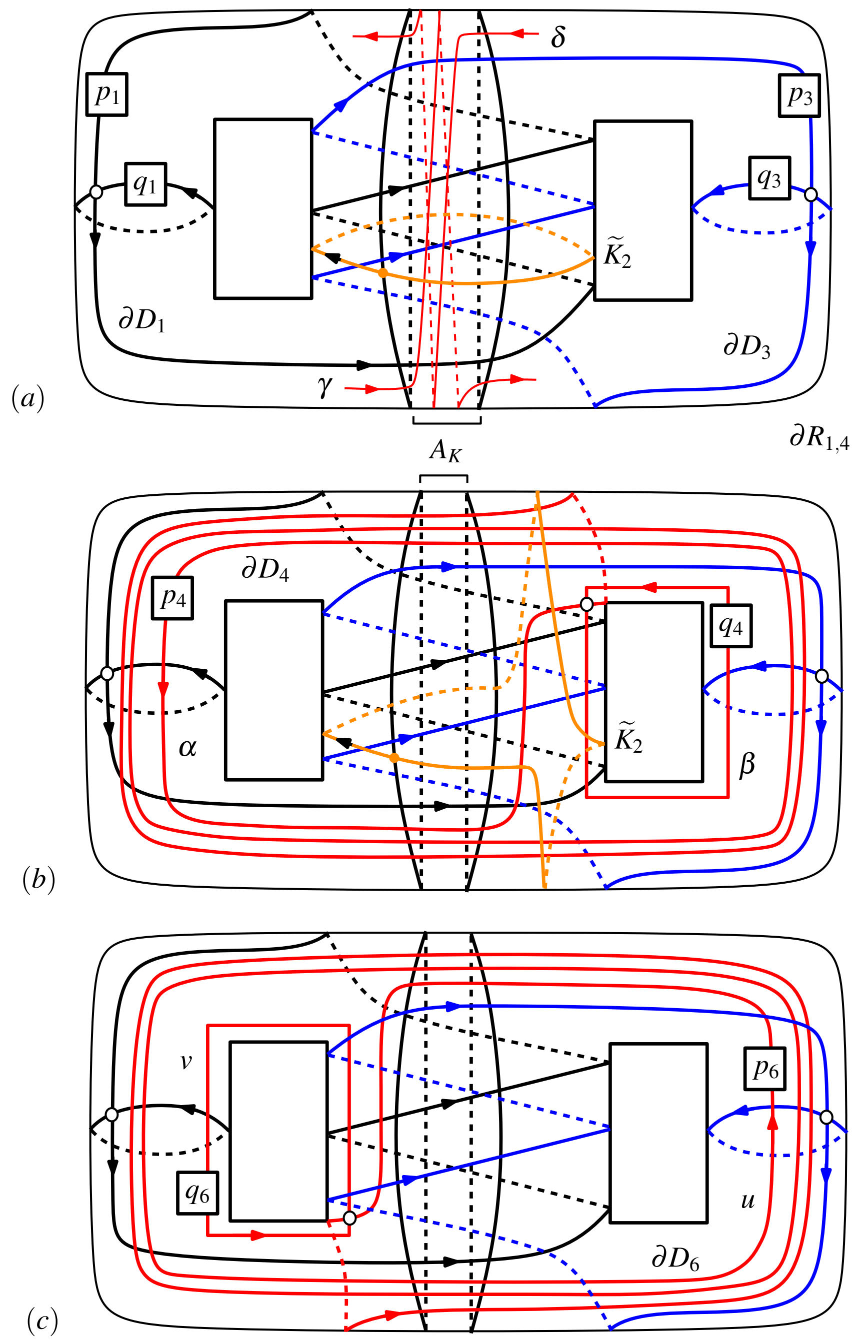

Figure 19: The type 1 Heegaard circles for $R_{1,4} \cup R_{4,1}$: (a) $\partial D_1 \sqcup \partial D_3$, (b) ∂D_4, and (c) ∂D_6 ($n = 0$, $p_2 = 2$, $p_5 = 3$).

7.4.2 Fundamental group presentations II

Recall that $D_1, D_3 \subset R_{1,4}$ are the compression disks of $\partial R_{1,4} \setminus \omega_3'$ and $\partial R_{1,4} \setminus \omega_1$, respectively. Therefore we have that $\pi_1(R_{1,4}) = \langle x_1, x_3 \mid - \rangle$ where the free generators x_1, x_3 represent the circles in $R_{1,4}$ dual to the disks D_1 and D_3 constructed in §7.4.1, respectively; similarly, $\pi_1(R_{4,1}) = \langle x_4, x_6 \mid - \rangle$ where x_4, x_6 represent the circles in $R_{4,1}$ dual to the disks D_4 and D_6, respectively.

Set $\varepsilon_i = q_i - p_i$ for $i = 1, 3, 4, 6$.

Lemma 7.11. $\gcd(p_i, \varepsilon_i) = 1$ *for* $i = 1, 3, 4, 6$*, and* $\varepsilon_i = q_i - p_i \in \{\pm 1\}$ *for* $i = 4, 6$.

Proof. That $\gcd(p_i, \varepsilon_i) = 1$ follows from the fact that $\gcd(p_i, q_i) = 1$.

From Fig. 18 we have that, in $\pi_1(R_{4,1}) = \langle x_4, x_6 \mid - \rangle$,

$$\omega_1 = (x_4^{p_4} x_6^{p_6})^{p_5 - 1} x_4^{p_4} x_6^{q_6} \quad \text{and} \quad \omega_3' = (x_6^{p_6} x_4^{p_4})^{p_5 - 1} x_6^{p_6} x_4^{q_4}$$

relative to the base points $\omega_1 \cap \omega_6'$ and $\omega_3' \cap \omega_4$. By Proposition 7.5, ω_3' is a Seifert circle in $R_{4,1}$ disjoint from the power circle $\omega_6' \subset T_1 \subset R_{4,1}$. Therefore, by Lemmas 6.8(1)(d) and 7.9 the word $(x_6 x_4^{p_4})^{p_5 - 1} x_6 x_4^{q_4}$ obtained by replacing $x_6^{p_6}$ with x_6 in the word that represents ω_3' must be primitive in the free group $\langle x_4, x_6 \mid - \rangle$. Since $p_4 \geq 2$, by Lemma 7.10 we must have $q_4 = p_4 \pm 1$, hence that $\varepsilon_4 \in \{\pm 1\}$. That $\varepsilon_6 \in \{\pm 1\}$ follows in a similar way by considering the word for ω_1. □

For convenience, in the sequel we will denote the generators x_1, x_3 of the free group $\pi_1(R_{1,4}) = \langle x_1, x_3 \mid - \rangle$ and their inverses by x, y and X, Y, respectively.

Let $\mathbb{F}_2 = \langle x, y \mid - \rangle$ denote the rank two free group generated by x, y and $\mathbb{M}_2$ the monoid generated by x, X, y, Y. Denote the cyclic reduction of any word $w \in \mathbb{F}_2$ by $[\![w]\!]$. For any two words w_1, w_2 in the monoid $\mathbb{M}_2$ we denote their equality in $\mathbb{M}_2$ by $w_1 \cong w_2$ and in the free group $\mathbb{F}_2$ by $w_1 = w_2$. Thus $w_1 \cong w_2$ implies that $w_1 = w_2$, and $x^2 y \cong xxy \cong [\![Xx^3 y]\!]$ but $x^2 y \not\cong Xx^3 y \not\cong x^3 X y$.

Cyclic permutations of a word w in $\mathbb{F}_2$ are performed by treating w as an element in $\mathbb{M}_2$, that is, without performing any cancellations on w.

For any two words w_1, w_2 in $\mathbb{F}_2$, we say that

- w_1 is equivalent to w_2 and write $w_1 \equiv w_2$ if w_2 is some cyclic permutation of w_1 or w_1^{-1}.
- w_1 divides w_2 if $w_2 \cong a \cdot w_1 \cdot b$ for some (possibly empty) words a, b.
- $w_1 \,\|\, w_2$ if there is a word u such that $[\![w_1]\!] \equiv u$ and $[\![w_2]\!] \equiv u \cdot v$, that is, if some word equivalent to $[\![w_1]\!]$ divides some word equivalent to $[\![w_2]\!]$.

With this notation the following result follows from Kaneto's theorem [15]:

Lemma 7.12. ([15, Theorem 1]) *If $\langle x,y \mid r_1,r_2 \rangle$ is a presentation of $\pi_1(\mathbb{S}^3)$ obtained from a genus two Heegaard splitting of $\mathbb{S}^3$ then, for some $\{i,j\}=\{1,2\}$, either $[\![r_i]\!]\equiv x$ and $[\![r_j]\!]\equiv y$, or $r_i \,\|\, r_j$.* $\square$

Unlike the division relation, the relation $\|$ is not transitive: if $w_1 = x^2y$, $w_2 = x^2y^2$, $w_3 = xy^2xY$ then $w_1 \,\|\, w_2$ and $w_2 \equiv xy^2x \,\|\, w_3$; however, none of the cyclic permutations x^2y, xyx, yx^2 of w_1 divides any of the cyclic permutations xy^2xY, y^2xYx, $yxYxy$, $xYxy^2$, Yxy^2x of w_3, from which it follows that $w_1 \not\| \, w_3$. We have however the following restricted version of transitivity for $\|$.

Lemma 7.13. *Suppose that w_1 and w_2 are cyclically reduced words in $\mathbb{F}_2$ with $w_1 \,\|\, w_2$. If each cyclic permutation of w_1 is divisible by one of the words $s,t \in \mathbb{F}_2$ then $s \,\|\, w_2$ or $t \,\|\, w_2$.*

Proof. Without loss of generality we may assume that $w_2 \cong u \cdot v$ for some cyclic permutation u of w_1, and that s divides u; by definition it follows that $s \,\|\, w_2$. $\square$

7.4.3 Presentations for the group $\pi_1(R_{1,4} \cup_\partial R_{4,1})$

In order to apply Lemma 7.12 to the group presentation

$$\pi_1(R_{1,4} \cup_\partial R_{4,1}) = \langle x,y \mid \partial D_4, \partial D_6 \rangle$$

we need to determine the words represented by the circles $\partial D_4, \partial D_6 \subset \partial R_{4,1} = \partial R_{1,4}$ in the free group $\pi_1(R_{1,4}) = \langle x,y \mid - \rangle$. At this point we remind the reader that the bound $p_i \geq 2$ holds for each $1 \leq i \leq 6$.

We shall see below that some of the circles representing ∂D_4 or ∂D_6 contain disjoint parallel copies of the oriented arcs γ and δ shown in Fig. 19(a), obtained by Dehn-twisting once a corresponding spanning arc in the annulus $A_K \subset \partial R_{1,4}$ in the indicated directions. Reading the oriented intersections of γ and δ with the disks $D_1, D_3 \subset R_{1,4}$ produces the words

$$\gamma = (x^{p_1}y^{p_3})^{p_2} \cdot (X^{p_1}Y^{p_3})^{p_2} \quad \text{and} \quad \delta = (Y^{p_3}X^{p_1})^{p_2} \cdot (y^{p_3}x^{p_1})^{p_2},$$

which will appear as factors in some of the words for $\partial D_4, \partial D_6 \in \pi_1(R_{1,4}) = \langle x,y \mid - \rangle$.

Let α and β be oriented components of the collections with p_4 and q_4 circles shown in Fig. 19(b), respectively, so that homologically we have $\partial D_4 = p_4\alpha + q_4\beta$ and $\beta = -\omega_4$. It follows that $\partial D_4 = w_4(\alpha,\beta)$ in $\pi_1(R_{1,4}) = \langle x,y \mid - \rangle$, where $w_4(\alpha,\beta)$ is a cyclically reduced primitive word in the free group $\langle \alpha,\beta \mid - \rangle$ (which is unique up to cyclic order) with abelianization $p_4\alpha + q_4\beta$. Since we have by Lemma 7.11 that $q_4 = p_4 + \varepsilon_4$ with $\varepsilon_4 = \pm 1$, we can take $\partial D_4 = \partial D_4^+ = (\alpha\beta)^{p_4} \cdot \beta$ if $\varepsilon_4 = +1$ and $\partial D_4 = \partial D_4^- = \alpha \cdot (\alpha\beta)^{p_4-1}$ if $\varepsilon_4 = -1$.

In a similar way, in $\pi_1(R_{1,4})$, we have $\partial D_6 = \partial D_6^+ = (uv)^{p_6} \cdot v$ if $\varepsilon_6 = +1$ and $\partial D_6 = \partial D_6^- = u \cdot (uv)^{p_6-1}$ if $\varepsilon_6 = -1$, where u, v are oriented components of the collections in Fig. 19(c) with p_6 and q_6 circles, respectively, so that $v = \omega_6'$.

Taking $\alpha \cap \beta$ and $u \cap v$ as base points, the words corresponding to α, β and u, v in $\pi_1(R_{1,4})$ after $n \in \mathbb{Z}$ Dehn twists along the annulus A_K, with $n > 0$ taken as the direction indicated by the arrows on the arcs γ, δ in Fig. 19(a), are given by the following expressions obtained with the convention that α, say, reads x whenever it intersects the oriented circle ∂D_1 from right-to-left, and $x^{-1} = X$ otherwise:

$\alpha = [\delta^n x^{q_1-p_1} \gamma^n y^{p_3-q_3}]^{p_5-1} \delta^n x^{q_1-p_1} \gamma^n x^{p_1} [y^{p_3} x^{p_1}]^{p_2-1}$
$\quad = [\delta^n x^{\varepsilon_1} \gamma^n Y^{\varepsilon_3}]^{p_5-1} \delta^n x^{\varepsilon_1} \gamma^n x^{p_1} [y^{p_3} x^{p_1}]^{p_2-1}$,
$\beta = \omega_4^{-1} = (X^{p_1} Y^{p_3})^{p_2-1} X^{p_1} Y^{p_3+\varepsilon_3} = (X^{p_1} Y^{p_3})^{p_2} Y^{\varepsilon_3}$,
$\alpha\beta = [\delta^n x^{\varepsilon_1} \gamma^n Y^{\varepsilon_3}]^{p_5} Y^{p_3}$,
$u = [\gamma^n y^{p_3-q_3} \delta^n x^{q_1-p_1}]^{p_5-1} \gamma^n y^{p_3-q_3} \delta^n Y^{p_3} (X^{p_1} Y^{p_3})^{p_2-1}$
$\quad = [\gamma^n Y^{\varepsilon_3} \delta^n x^{\varepsilon_1}]^{p_5-1} \gamma^n Y^{\varepsilon_3} \delta^n Y^{p_3} (X^{p_1} Y^{p_3})^{p_2-1}$,
$v = \omega_1 = (y^{p_3} x^{p_1})^{p_2-1} y^{p_3} x^{p_1+\varepsilon_1} = (y^{p_3} x^{p_1})^{p_2} x^{\varepsilon_1}$,
$uv = [\gamma^n Y^{\varepsilon_3} \delta^n x^{\varepsilon_1}]^{p_5} x^{p_1}$.

Therefore we obtain the following words for ∂D_4 and ∂D_6:

$$
\begin{aligned}
\partial D_4^+ &= (\alpha\beta)^{p_4}\beta = \Big[[\delta^n x^{\varepsilon_1} \gamma^n Y^{\varepsilon_3}]^{p_5} Y^{p_3}\Big]^{p_4} (X^{p_1} Y^{p_3})^{p_2} Y^{\varepsilon_3} \\
\partial D_4^- &= \alpha(\alpha\beta)^{p_4-1} \\
&= [\delta^n x^{\varepsilon_1} \gamma^n Y^{\varepsilon_3}]^{p_5-1} \delta^n x^{\varepsilon_1} \gamma^n x^{p_1} [y^{p_3} x^{p_1}]^{p_2-1} \Big[[\delta^n x^{\varepsilon_1} \gamma^n Y^{\varepsilon_3}]^{p_5} Y^{p_3}\Big]^{p_4-1}
\end{aligned}
$$

$$
\begin{aligned}
\partial D_6^+ &= (uv)^{p_6} v = \Big[[\gamma^n Y^{\varepsilon_3} \delta^n x^{\varepsilon_1}]^{p_5} x^{p_1}\Big]^{p_6} (y^{p_3} x^{p_1})^{p_2} x^{\varepsilon_1}, \\
\partial D_6^- &= u(uv)^{p_6-1} \\
&= [\gamma^n Y^{\varepsilon_3} \delta^n x^{\varepsilon_1}]^{p_5-1} \gamma^n Y^{\varepsilon_3} \delta^n Y^{p_3} (X^{p_1} Y^{p_3})^{p_2-1} \Big[[\gamma^n Y^{\varepsilon_3} \delta^n x^{\varepsilon_1}]^{p_5} x^{p_1}\Big]^{p_6-1}
\end{aligned}
$$

There are 3 cases to consider, depending on the value of $n \in \mathbb{Z}$.

7.4.4 The case $n = 0$

We have the identities

$$
\begin{aligned}
\partial D_4^+ &= [\underbrace{(x^{\varepsilon_1} Y^{\varepsilon_3})^{p_5} Y^{p_3}}_{s}]^{p_4} \cdot (X^{p_1} Y^{p_3})^{p_2} Y^{\varepsilon_3} \\
\partial D_4^- &= (x^{\varepsilon_1} Y^{\varepsilon_3})^{p_5-1} \underbrace{x^{p_1+\varepsilon_1} (y^{p_3} x^{p_1})^{p_2-1} \cdot [(x^{\varepsilon_1}}_{s} \underbrace{Y^{\varepsilon_3})^{p_5} Y^{p_3}}_{t}]^{p_4-1} \\
\partial D_6^+ &= [(Y^{\varepsilon_3} x^{\varepsilon_1})^{p_5} x^{p_1}]^{p_6} \cdot (y^{p_3} x^{p_1})^{p_2} x^{\varepsilon_1} \\
\partial D_6^- &= (Y^{\varepsilon_3} x^{\varepsilon_1})^{p_5-1} Y^{p_3+\varepsilon_3} \cdot (X^{p_1} Y^{p_3})^{p_2-1} \cdot [(Y^{\varepsilon_3} x^{\varepsilon_1})^{p_5} x^{p_1}]^{p_6-1}
\end{aligned}
$$

It is then not hard to see that

1. any cyclic permutation of $[\![\partial D_4^+]\!]$ is divisible by $s = (x^{\varepsilon_1} Y^{\varepsilon_3})^{p_5} Y^{p_3}$ but $s \not\parallel \partial D_6^{\pm}$,
2. any cyclic permutation of $[\![\partial D_4^-]\!]$ is divisible by

$$s = x^{p_1+\varepsilon_1}(y^{p_3}x^{p_1})^{p_2-1}x^{\varepsilon_1} \quad \text{or} \quad t = Y^{\varepsilon_3}(x^{\varepsilon_1}Y^{\varepsilon_3})^{p_5-1}Y^{p_3}$$

but $s,t \not\parallel \partial D_6^{\pm}$.

Observe now that the words $\partial D_4^{\pm}$ and $\partial D_6^{\pm}$ are related by the following symmetry:

(S) For each $* \in \{\pm\}$ there is a word $w(x,y,X,Y;a,b,c,d,e)$ in the free group $\langle x,y,X,Y \mid - \rangle$ depending on parameters $a,b,c,d,e \in \mathbb{Z}$ such that

$$\partial D_4^* = w(x,y,X,Y;\varepsilon_1,p_1,\varepsilon_3,p_3,p_4),$$
$$\partial D_6^* = w(Y,X,y,x;\varepsilon_3,p_3,\varepsilon_1,p_1,p_6).$$

Similarly, in items (1) and (2) the words for s and t are each of the form

$$W(x,y,X,Y;\varepsilon_1,p_1,\varepsilon_3,p_3),$$

that is, independent of p_4 and p_6. Therefore, replacing s and t in (1) and (2) above with the words s',t' corresponding to the transformation

$$W(x,y,X,Y;\varepsilon_1,p_1,\varepsilon_3,p_3) \mapsto W(Y,X,y,x;\varepsilon_3,p_3,\varepsilon_1,p_1)$$

and using the symmetry (S) above, statements (1) and (2) transform into the following equivalent statements:

($1'$) any cyclic permutation of $[\![\partial D_6^+]\!]$ is divisible by $s' = (Y^{\varepsilon_3} x^{\varepsilon_1})^{p_5} x^{p_1}$ but $s' \not\parallel \partial D_4^{\pm}$,

($2'$) any cyclic permutation of $[\![\partial D_6^-]\!]$ is divisible by either

$$s' = Y^{p_3+\varepsilon_3}(X^{p_1}Y^{p_3})^{p_2-1}Y^{\varepsilon_3} \quad \text{or} \quad t' = x^{\varepsilon_1}(Y^{\varepsilon_3}x^{\varepsilon_1})^{p_5-1}x^{p_1}$$

but $s',t' \not\parallel \partial D_4^{\pm}$.

We therefore have by Lemma 7.13 that $[\![\partial D_i^{\pm}]\!] \not\parallel [\![\partial D_j^{\pm}]\!]$ for each $\{i,j\} = \{1,2\}$.

In all cases that follow for type 1 or 2 Heegaard diagrams we will explicitly establish the equivalent version of statements (1) and (2) above, and that the corresponding equivalent versions of ($1'$) and ($2'$) also hold will follow by the argument above.

Remark 7.14. *The values $p_i = 2$ for $1 \le i \le 5$, $p_6 = 4$, $\varepsilon_1 = +1$ and $\varepsilon_i = -1$ for $i = 3,4,6$ produce an integral homology 3-sphere $R_{1,4} \cup_\partial R_{4,1}$ which by the argument above is not homeomorphic to $\mathbb{S}^3$ for $n = 0$. Thus in general integral homology does not differentiate the manifolds $R_{1,4} \cup_\partial R_{4,1}$ from $\mathbb{S}^3$.*

7.4.5 The case $n > 0$

$$\partial D_4^+ = \Bigg(\Bigg[\Big[\underbrace{(Y^{p_3}X^{p_1})^{p_2}}_{C}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}\underbrace{(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Bigg]^{p_5} Y^{p_3}}_{A}\Bigg)^{p_4} \cdot$$
$$\cdot \underbrace{(X^{p_1}Y^{p_3})^{p_2}Y^{\varepsilon_3}}_{B}$$

$$\partial D_4^- = \Bigg[\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Bigg]^{p_5-1} \cdot$$
$$\cdot \Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}\underbrace{(X^{p_1}Y^{p_3})^{p_2}\Big]^n x^{p_1}(y^{p_3}x^{p_1})^{p_2-1}}_{t} \cdot$$
$$\cdot \Bigg(\Bigg[\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Bigg]^{p_5} Y^{p_3}\Bigg)^{p_4-1}$$

$$\partial D_6^+ = \Bigg(\Bigg[\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Bigg]^{p_5} x^{p_1}\Bigg)^{p_6} \cdot$$
$$\cdot (y^{p_3}x^{p_1})^{p_2}x^{\varepsilon_1}$$

$$\partial D_6^- = \Bigg[\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Bigg]^{p_5-1} \cdot$$
$$\cdot \Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n Y^{p_3}(X^{p_1}Y^{p_3})^{p_2-1} \cdot$$
$$\cdot \Bigg(\Bigg[\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Bigg]^{p_5} x^{p_1}\Bigg)^{p_6-1}$$

1. Any cyclic permutation of $\llbracket \partial D_4^+ \rrbracket$ is divisible by $s = Y^{3p_3+\varepsilon_3}$ (located in AC) or
$$t = ABC = (X^{p_1}Y^{p_3})^{p_2}Y^{p_3+\varepsilon_3}(X^{p_1}Y^{p_3})^{p_2}Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2},$$
but $s,t \nparallel \partial D_6^{\pm}$.

2. Any cyclic permutation of $\llbracket \partial D_4^- \rrbracket$ is divisible by either
$$s = Y^{3p_3+\varepsilon_3} \quad \text{or} \quad t = (X^{p_1}Y^{p_3})^{p_2}x^{p_1}(y^{p_3}x^{p_1})^{p_2-1}$$
but $s,t \nparallel \partial D_6^{\pm}$.

7.4.6 The case $n < 0$

We use the identities

$$\gamma^n = (\gamma^{-1})^{|n|} = \Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}$$

$$\delta^n = (\delta^{-1})^{|n|} = \Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}$$

to obtain the words

$$\partial D_4^+ = \Bigg(\Bigg[\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}\underbrace{Y^{\varepsilon_3}\Bigg]^{p_5}Y^{p_3}}_{s}\Bigg)^{p_4}\cdot$$
$$\cdot\,(X^{p_1}Y^{p_3})^{p_2}Y^{\varepsilon_3}$$

$$\partial D_4^- = \Bigg[\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}Y^{\varepsilon_3}\Bigg]^{p_5-1}\cdot$$
$$\cdot\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\underbrace{\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}\cdot x^{p_1}(y^{p_3}x^{p_1})^{p_2-1}}_{A}\cdot$$
$$\cdot\Bigg(\Bigg[\Big[\underbrace{(X^{p_1}Y^{p_3})^{p_2}}_{B}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}\underbrace{Y^{\varepsilon_3}\Bigg]^{p_5}Y^{p_3}}_{s}\Bigg)^{p_4-1}$$

$$\partial D_6^+ = \Bigg(\Bigg[\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}Y^{\varepsilon_3}\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Bigg]^{p_5}x^{p_1}\Bigg)^{p_6}\cdot$$
$$\cdot\,(y^{p_3}x^{p_1})^{p_2}x^{\varepsilon_1}$$

$$\partial D_6^- = \Bigg[\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}Y^{\varepsilon_3}\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Bigg]^{p_5-1}\cdot$$
$$\cdot\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}Y^{\varepsilon_3}\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}Y^{p_3}(X^{p_1}Y^{p_3})^{p_2-1}\cdot$$
$$\cdot\Bigg(\Bigg[\Big[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\Big]^{|n|}Y^{\varepsilon_3}\Big[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\Big]^{|n|}x^{\varepsilon_1}\Bigg]^{p_5}x^{p_1}\Bigg)^{p_6-1}$$

1. Any cyclic permutation of $\llbracket \partial D_4^+ \rrbracket$ is divisible by $s = Y^{p_3+\varepsilon_3}$ but $s \not\parallel \partial D_6^{\pm}$.
2. Any cyclic permutation of $\llbracket \partial D_4^- \rrbracket$ is divisible by either

$$s = Y^{p_3+\varepsilon_3} \quad \text{or} \quad t = AB = (y^{p_3}x^{p_1})^{p_2} Y^{p_3} (X^{p_1}Y^{p_3})^{p_2}$$

but $s,t \not\parallel \partial D_6^{\pm}$.

By Lemmas 7.13 and 7.12, we have therefore established the following result:

Lemma 7.15. *If the Heegaard diagram of $R_{1,4} \cup_\partial R_{4,1}$ is of type 1 then $R_{1,4} \cup_\partial R_{4,1} \neq \mathbb{S}^3$.* □

7.5 The type 2 Heegaard diagrams for $R_{1,4} \cup_\partial R_{4,1}$

We follow the outline of the analysis of type 1 Heegaard diagrams given in §7.4.

The circles $\partial E^{(2)}_{5,6}, \partial D_4, \partial D_6 \subset \partial R_{4,1} = \partial R_{1,4}$ are shown in Fig. 20, and the circles $\partial D_i \subset \partial R_{1,4}$, $i = 1,3,4,6$, comprising the Heegaard diagram of type 2 for $R_{1,4} \cup_\partial R_{4,1}$ are shown in Fig. 21 (where $n = 0$, $p_2 = 2$, $p_5 = 3$).

The circles $\alpha, \beta\, u, v \subset \partial R_{1,4}$ are defined and their words in $\pi_1(R_{1,4}) = \langle x, y \mid - \rangle$ computed relative to the base points $\alpha \cap \beta$ and $u \cap v$ as in §7.4, obtaining the following identities:

$\alpha = (x^{p_1}y^{p_3})^{p_2}[\delta^n x^{\varepsilon_1}\gamma^n Y^{\varepsilon_3}]^{p_5} y^{p_3+\varepsilon_3}$, $\beta = Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}$

$\beta\alpha = [Y^{\varepsilon_3}\delta^n x^{\varepsilon_1}\gamma^n]^{p_5} y^{p_3}$

$u = (Y^{p_3}X^{p_1})^{p_2}[\gamma^n Y^{\varepsilon_3}\delta^n x^{\varepsilon_1}]^{p_5} X^{p_1+\varepsilon_1}$, $v = x^{q_1}y^{p_3}(x^{p_1}y^{p_3})^{p_2-1} = x^{\varepsilon_1}(x^{p_1}y^{p_3})^{p_2}$

$vu = [x^{\varepsilon_1}\gamma^n Y^{\varepsilon_3}\delta^n]^{p_5} X^{p_1}$

The conclusion of Lemma 7.11 applies in the present context, so we can take $\partial D_4 = \partial D_4^+ = (\beta\alpha)^{p_4}\beta$ if $\varepsilon_4 = +1$ and $\partial D_4 = \partial D_4^- = \alpha(\beta\alpha)^{p_4-1}$ if $\varepsilon_4 = -1$, and $\partial D_6 = \partial D_6^+ = (vu)^{p_6}v$ if $\varepsilon_6 = +1$ and $\partial D_6 = \partial D_6^- = u(vu)^{p_6-1}$ if $\varepsilon_6 = -1$. This yields the following words:

$$\partial D_4^+ = (\beta\alpha)^{p_4}\beta = \Big[[Y^{\varepsilon_3}\delta^n x^{\varepsilon_1}\gamma^n]^{p_5} y^{p_3}\Big]^{p_4} Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}$$

$$\partial D_4^- = \alpha(\beta\alpha)^{p_4-1} = (x^{p_1}y^{p_3})^{p_2}[\delta^n x^{\varepsilon_1}\gamma^n Y^{\varepsilon_3}]^{p_5} y^{p_3+\varepsilon_3}\Big[[Y^{\varepsilon_3}\delta^n x^{\varepsilon_1}\gamma^n]^{p_5} y^{p_3}\Big]^{p_4-1}$$

$$\partial D_6^+ = (vu)^{p_6}v = \Big[[x^{\varepsilon_1}\gamma^n Y^{\varepsilon_3}\delta^n]^{p_5} X^{p_1}\Big]^{p_6} x^{\varepsilon_1}(x^{p_1}y^{p_3})^{p_2}$$

$$\partial D_6^- = u(vu)^{p_6-1} = (Y^{p_3}X^{p_1})^{p_2}[\gamma^n Y^{\varepsilon_3}\delta^n x^{\varepsilon_1}]^{p_5} X^{p_1+\varepsilon_1}\Big[[x^{\varepsilon_1}\gamma^n Y^{\varepsilon_3}\delta^n]^{p_5} X^{p_1}\Big]^{p_6-1}$$

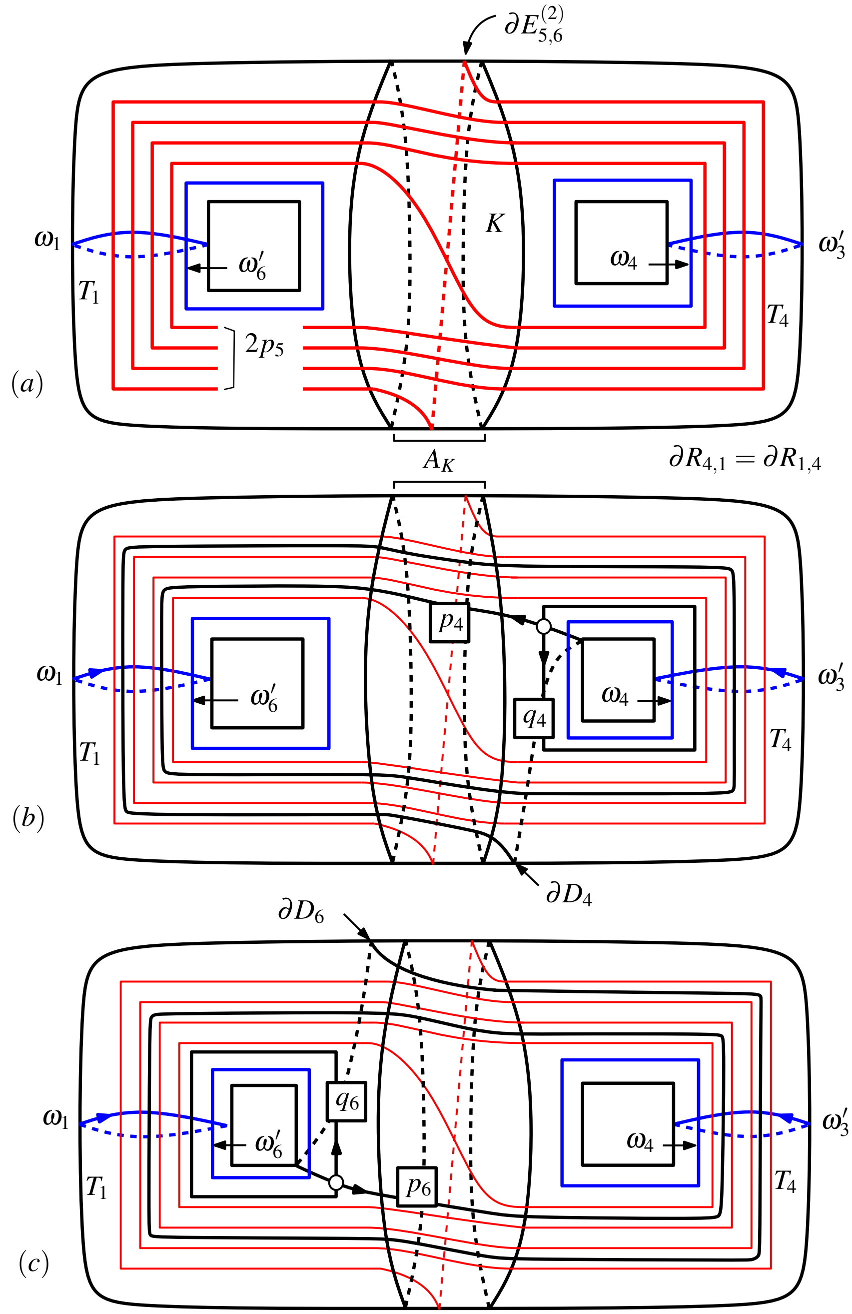

Figure 20: The circles $\partial E_{5,6}^{(2)}$ ($n=0$, $p_5=2$) and $\partial D_4, \partial D_6$ in $\partial R_{4,1} = \partial R_{1,4}$.

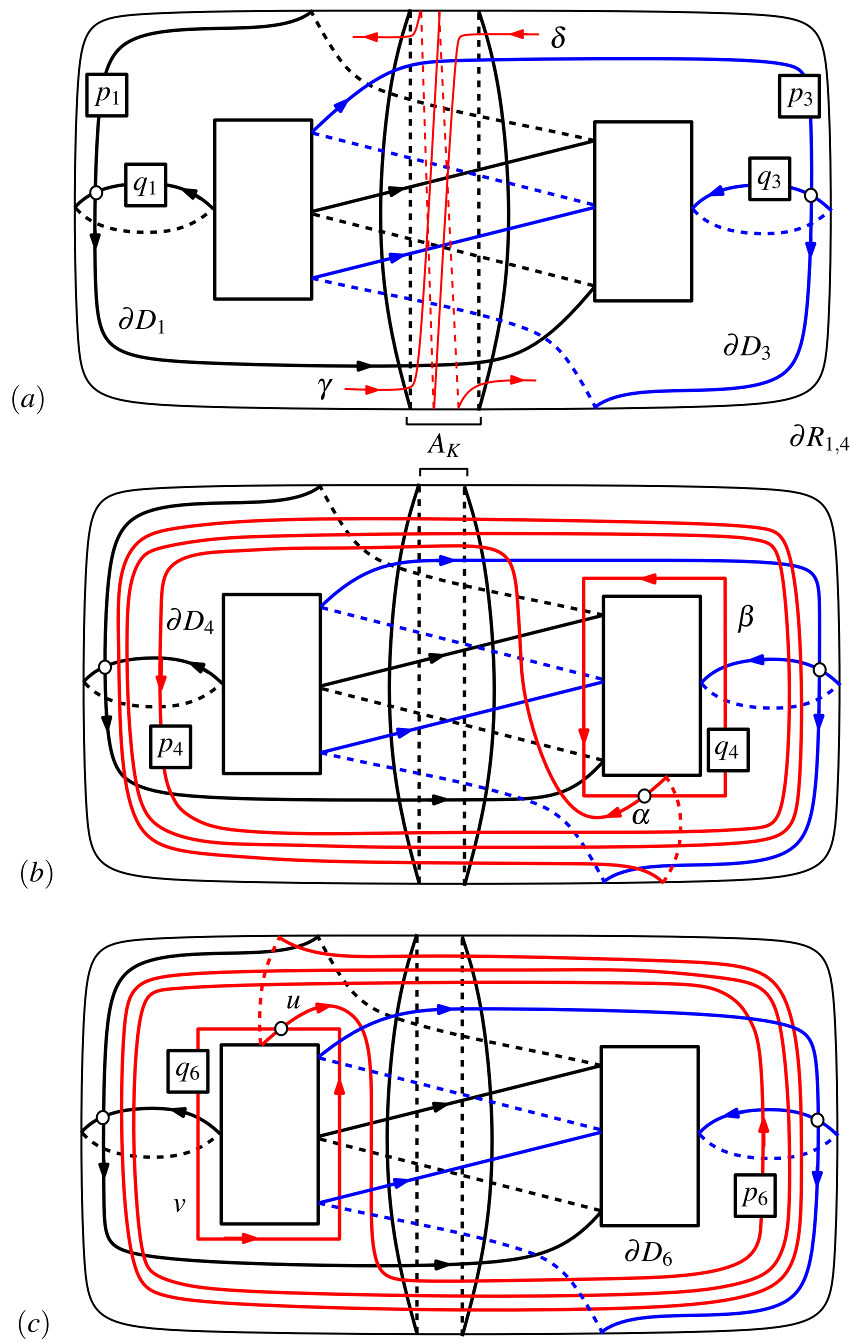

Figure 21: The type 2 Heegaard circles for $R_{1,4} \cup_\partial R_{4,1}$ ($n = 0$, $p_2 = 2$, $p_5 = 3$).

7.5.1 The case $n = 0$

$$
\begin{aligned}
\partial D_4^+ &= \Big[(Y^{\varepsilon_3}x^{\varepsilon_1})^{p_5}y^{p_3}\Big]^{p_4}\cdot Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}\\
&= (\underbrace{Y^{\varepsilon_3}x^{\varepsilon_1}}_{B})^{p_5}\,\underbrace{y^{p_3}\Big[(Y^{\varepsilon_3}}_{s}\,\underbrace{x^{\varepsilon_1})^{p_5}y^{p_3}\Big]^{p_4-1}\cdot Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}}_{A}
\end{aligned}
$$

$$
\begin{aligned}
\partial D_4^- &= (x^{p_1}y^{p_3})^{p_2}(x^{\varepsilon_1}\,\underbrace{Y^{\varepsilon_3})^{p_5}y^{p_3+\varepsilon_3}}_{s}\Big[(Y^{\varepsilon_3}x^{\varepsilon_1})^{p_5}y^{p_3}\Big]^{p_4-1}\\
\partial D_6^+ &= \Big[(x^{\varepsilon_1}Y^{\varepsilon_3})^{p_5}X^{p_1}\Big]^{p_6}x^{\varepsilon_1}(x^{p_1}y^{p_3})^{p_2}\\
\partial D_6^- &= (Y^{p_3}X^{p_1})^{p_2}(Y^{\varepsilon_3}x^{\varepsilon_1})^{p_5}X^{p_1+\varepsilon_1}\Big[(x^{\varepsilon_1}Y^{\varepsilon_3})^{p_5}X^{p_1}\Big]^{p_6-1}
\end{aligned}
$$

1. Any cyclic permutation of $[\![\partial D_4^+]\!]$ is divisible by

   $$s = y^{p_3-\varepsilon_3} \quad \text{or} \quad t = AB = x^{\varepsilon_1}Y^{\varepsilon_3}X^{p_1}(Y^{p_3}X^{p_1})^{p_2-1}Y^{\varepsilon_3}x^{\varepsilon_1}$$

   but $s,t \not\parallel \partial D_6^{\pm}$.

2. Any cyclic permutation of $[\![\partial D_4^-]\!]$ is divisible by $s = y^{p_3-\varepsilon_3}$ or $t = Y^{\varepsilon_3}x^{\varepsilon_1}y^{p_3}$ but $s,t \not\parallel \partial D_6^{\pm}$.

7.5.2 The case $n > 0$

$$
\partial D_4^+ = \Bigg(\bigg[Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}\,\underbrace{x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}}_{s}\,y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n\bigg]^{p_5}y^{p_3}\Bigg)^{p_4}\cdot
$$
$$
\cdot Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}
$$

$$
\partial D_4^- = (x^{p_1}y^{p_3})^{p_2}\bigg[\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}\,\underbrace{x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}}_{s}\,y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\bigg]^{p_5}\cdot
$$
$$
\cdot y^{p_3+\varepsilon_3}\Bigg(\bigg[Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n\bigg]^{p_5}y^{p_3}\Bigg)^{p_4-1}
$$

$$
\partial D_6^+ = \Bigg(\bigg[x^{\varepsilon_1}\Big[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\Big]^n Y^{\varepsilon_3}\Big[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\Big]^n\bigg]^{p_5}X^{p_1}\Bigg)^{p_6}\cdot
$$
$$
\cdot x^{\varepsilon_1}(x^{p_1}y^{p_3})^{p_2}
$$

$$\partial D_6^- = (Y^{p_3}X^{p_1})^{p_2}\left[\left[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\right]^n Y^{\varepsilon_3}\left[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\right]^n x^{\varepsilon_1}\right]^{p_5}\cdot$$

$$\cdot X^{p_1+\varepsilon_1}\left(\left[x^{\varepsilon_1}\left[(x^{p_1}y^{p_3})^{p_2}(X^{p_1}Y^{p_3})^{p_2}\right]^n Y^{\varepsilon_3}\left[(Y^{p_3}X^{p_1})^{p_2}(y^{p_3}x^{p_1})^{p_2}\right]^n\right]^{p_5} X^{p_1}\right)^{p_6-1}$$

In this case we have that any cyclic permutation of $[\![\partial D_4^\pm]\!]$ is divisible by $s = x^{2p_1+\varepsilon_1}$ (located in several disjoint sites) but $s \not\parallel \partial D_6^\pm$.

7.5.3 The case $n < 0$

$$\partial D_4^+ = \left(\left[\underbrace{Y^{\varepsilon_3}}_{C}\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|} x^{\varepsilon_1}\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|}\right]^{p_5}\underbrace{y^{p_3}}_{A}\right)^{p_4}\cdot$$

$$\cdot\underbrace{Y^{\varepsilon_3}(Y^{p_3}X^{p_1})^{p_2}}_{B}$$

$$\partial D_4^- = \underbrace{(x^{p_1}y^{p_3})^{p_2}}_{D}\left[\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|} x^{\varepsilon_1}\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|}\underbrace{Y^{\varepsilon_3}}_{A}\right]^{p_5}\cdot$$

$$\cdot\underbrace{y^{p_3+\varepsilon_3}}_{A'}\left(\left[\underbrace{Y^{\varepsilon_3}}_{B}\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|} x^{\varepsilon_1}\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|}\right]^{p_5}\underbrace{y^{p_3}}_{C}\right)^{p_4-1}$$

$$\partial D_6^+ = \left(\left[x^{\varepsilon_1}\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|} Y^{\varepsilon_3}\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|}\right]^{p_5} X^{p_1}\right)^{p_6}\cdot$$

$$\cdot x^{\varepsilon_1}(x^{p_1}y^{p_3})^{p_2}$$

$$\partial D_6^- = (Y^{p_3}X^{p_1})^{p_2}\left[\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|} Y^{\varepsilon_3}\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|} x^{\varepsilon_1}\right]^{p_5}\cdot$$

$$\cdot X^{p_1+\varepsilon_1}\left(\left[x^{\varepsilon_1}\left[(y^{p_3}x^{p_1})^{p_2}(Y^{p_3}X^{p_1})^{p_2}\right]^{|n|} Y^{\varepsilon_3}\left[(X^{p_1}Y^{p_3})^{p_2}(x^{p_1}y^{p_3})^{p_2}\right]^{|n|}\right]^{p_5} X^{p_1}\right)^{p_6-1}$$

1. Any cyclic permutation of $\llbracket \partial D_4^+ \rrbracket$ is divisible by either

$$s = AC = y^{p_3-\varepsilon_3} \quad \text{or} \quad t = ABC = Y^{\varepsilon_3}(X^{p_1}Y^{p_3})^{p_2-1}X^{p_1}Y^{\varepsilon_3}$$

but $s,t \not\parallel \partial D_6^{\pm}$.

2. Any cyclic permutation of $\llbracket \partial D_4^- \rrbracket$ is divisible by

$$s = AA'B = y^{p_3-\varepsilon_3} \quad \text{or} \quad t = CD = y^{p_3}(x^{p_1}y^{p_3})^{p_2}$$

but $s,t \not\parallel \partial D_6^{\pm}$.

By Lemmas 7.13 and 7.12, we have therefore established the following result:

Lemma 7.16. *If the Heegaard diagram of $R_{1,4} \cup_\partial R_{4,1}$ is of type 2 then $R_{1,4} \cup_\partial R_{4,1} \neq \mathbb{S}^3$.* □

We are now ready to give the proof of the first main theorem of this paper:

Proof of Theorem 1: Let $K \subset \mathbb{S}^3$ be a genus one hyperbolic knot and $\mathbb{T} = T_1 \sqcup \cdots \sqcup T_N$ a collection of $N \geq 1$ disjoint, mutually non-parallel once-punctured tori in X_K. By Lemma 4.3 we then have that $N \leq 6$, and if $N = 6$ then by Lemma 7.8 $\mathbb{S}^3$ has a genus two Heegaard splitting of type 1 or 2, contradicting Lemmas 7.15 and 7.16. Therefore $N \leq 5$. □

8 Examples of genus one hyperbolic knots in $\mathbb{S}^3$

By Lemmas 4.1 and 5.1, if $K \subset \mathbb{S}^3$ is a hyperbolic knot with a collection $\mathbb{T} \subset X_K$ of once-punctured tori then each complementary region of $\mathbb{T}$ is atoroidal and no circle in any component $T_i \subset \mathbb{T}$ has a companion annulus in X_K on either side of T_i. The next result shows that these two properties essentially characterize genus one hyperbolic knots and gives properties of some of its surgery manifolds. For notation, a surface S properly embedded in a manifold M is *strongly knotted* if the manifold obtained by cutting M along S is irreducible and boundary irreducible. As usual, $J \subset \partial X_K$ denotes the slope of the standard longitude of K.

Lemma 8.1. *Let $K \subset \mathbb{S}^3$ be a genus one knot whose exterior X_K contains a collection $\mathbb{T} = T_1 \sqcup \cdots \sqcup T_N \subset X_K$ of $N \geq 1$ mutually disjoint and non-parallel once-punctured tori.*

1. *If for each $1 \leq i \leq N$ the region $R_{i,i+1}$ is atoroidal and no circle in T_i has companion annuli in X_K on both sides of T_i then either K is a hyperbolic knot or $N = 1$ and K is the trefoil knot.*

2. *For K a hyperbolic knot and $r \subset \partial X_K$ any slope such that $\Delta(r,J) \geq 2$,*

   (a) *if some component $T_i \subset \mathbb{T}$ is strongly knotted then the manifold $X_K(r)$ is Haken,*

   (b) *if $N \geq 4$ then each component of $\mathbb{T}$ is strongly knotted and the manifold $X_K(r)$ is Haken and hyperbolic.*

Proof. For part (1), the hypotheses on the regions $R_{i,i+1}$ imply that any essential torus $T \subset X_K$ can be isotoped so as to intersect $\mathbb{T}$ minimally with $T \cap T_1$, say, a non-empty collection of circles which are non-trivial and mutually parallel in T and T_i.

For $R_{1,1} = \mathrm{cl}\,(X_K \setminus T_1 \times [-1,1])$, each component of $T \cap R_{1,1}$ is therefore an annulus which is either (a) a companion annulus in $R_{1,1}$ for one of the slopes $T \cap (T_1 \times \{-1,1\})$ or (b) a non-separating annulus in $R_{1,1}$ with one boundary component in each of $T_1 \times \{-1\}$ and $T_1 \times \{1\}$. By hypothesis not all the annuli in $T \cap R_{1,1}$ can be of type (a), while any annulus component of type (b) can be extended via an annulus in $T_1 \times [-1,1]$ to form a closed Klein bottle or non-separating torus in $X_K \subset \mathbb{S}^3$, which is impossible. Therefore K is not a satellite knot, so by [17] K is either a hyperbolic or torus knot, and in the latter case K must be the trefoil knot and $N = 1$.

For part (2)(a), assume for definiteness that T_1 is strongly knotted. Let $F = \partial R_{1,1} \subset X_K$ and let $r \subset \partial X_K$ be a slope with $\Delta(r,J) \geq 2$. If $X_K(r) = X_K \cup_\partial V_r$, where V_r is a solid torus and r bounds a disk in V_r, then the annulus $A = N(T_1) \cap \partial X_K$ is incompressible in the manifold $M = N(T_1) \cup_A V_r$, and we can write

$$X_K(r) = [R_{1,1} \cup N(T_1)] \cup V_r = R_{1,1} \cup_F [N(T_1) \cup_A V_r] = R_{1,1} \cup_F M.$$

Since $N(T_1) \approx T_1 \times [-1,1]$ with T_1 corresponding to $T_1 \times \{0\}$ and A to $(\partial T_1) \times [-1,1]$, if $D \subset M$ is a compression disk for $\partial M = F$ then the minimal intersection of A and D in M is nonempty, with $A \cap D \subset A$ consisting of a collection of spanning arcs of A. Hence if $E \subset D$ is an outermost disk cut out by an outermost arc of $A \cap D \subset D$ then E lies in $N(T_1)$ or V_r and ∂E intersects the core J of A minimally in one point, which is impossible since J, the core of A, runs $\Delta(r,J) \geq 2$ times around V_r and separates $\partial N(T_1)$. Therefore M is irreducible and boundary irreducible and so the manifold $X_K(r) = R_{1,1} \cup_F M$ is Haken.

For part (2)(b), suppose that $N \geq 4$ and there is an incompressible torus $\widehat{T}$ in $X_K(r)$. Since the manifold $R_{i,i}$ contains the collection $\mathbb{T} \setminus T_1$ of $N-1 \geq 3$ once-punctured tori, the once-punctured torus T_1 is strongly knotted by Lemmas 3.9 and 4.1.

After an isotopy, $\widehat{T}$ may be assumed to intersect V_r minimally in a non-empty collection of meridian disks, so that $T = \widehat{T} \cap X_K$ is an essential punctured torus

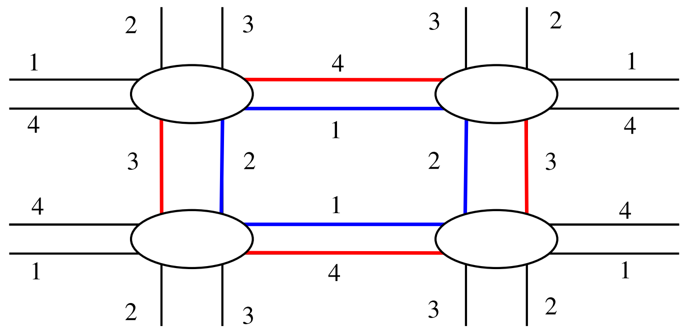

Figure 22: The graph $G_T = T \cap \mathbb{T} \subset T$.

which intersects $\mathbb{T}$ minimally in essential graphs $G_T = T \cap \mathbb{T} \subset T$ and $G = T \cap \mathbb{T} \subset \mathbb{T}$.

If $p = \Delta(r, J) \geq 2$ then each vertex of G_T has degree $pN \geq 8$ and so, by the initial part of Lemma 4.1 and by Lemma 4.2, both of which hold with T in place of the many punctured 2-sphere Q, for the reduced graph $\overline{G}_T$ (see §2.3) each of its edges has size at most 2, so each of its $V = |\partial T| \geq 2$ vertices has degree at least $pN/2 \geq 4$, and each of its $d \geq 0$ disk faces has at least 4 edges. Applying Euler's relation to the reduced graph $\overline{G}_T$ yields the relations

$$4V \leq 2E \leq 2V + 2d \implies V \leq d,$$
$$4d \leq 2E \leq 2V + 2d \implies d \leq V,$$

which imply that $d = V$, hence that $p = 2$ and $N = 4$, and that in $\overline{G}_T$ all vertices have degree 4, all faces are 4-sided disk faces, and each edge $\bar{e}$ is the amalgamation of two mutually parallel edges from G_T.

So if f is a 4-sided disk face of G_T that lies in, say, the region $R_{1,2}$, then the union of f and the bigon disk faces of G_T incident to each edge around f forms a 4-sided disk face $f^{3,4}$ of the graph $G_T^{3,4} = T \cap (T_3 \sqcup T_4) \subset T$ which lies in the region $R_{4,3} \supset R_{1,2}$ (see Fig. 22). Thus by Lemmas 2.1(3) and 4.1(2) the region $R_{4,3}$ is a genus two handlebody such that the disk $f^{3,4} \subset R_{4,3}$ intersects K minimally in 4 points. By Lemma 6.1, $(R_{4,3}, K)$ must be a simple or double pair, contradicting Lemma 6.8(2)(a) since the punctured tori $T_1, T_2 \subset R_{4,3}$ are neither boundary parallel nor mutually parallel in $R_{4,3}$. Therefore the Haken manifold $X_K(r)$ is atoroidal, hence hyperbolic by Thurston's hyperbolization theorem [17, 18]. □

The type 1 Heegaard diagrams for the manifold $M = R_{1,4} \cup_\partial R_{4,1}$ constructed in Section 7.4 can be adapted to yield knots in M that bound 5 mutually disjoint

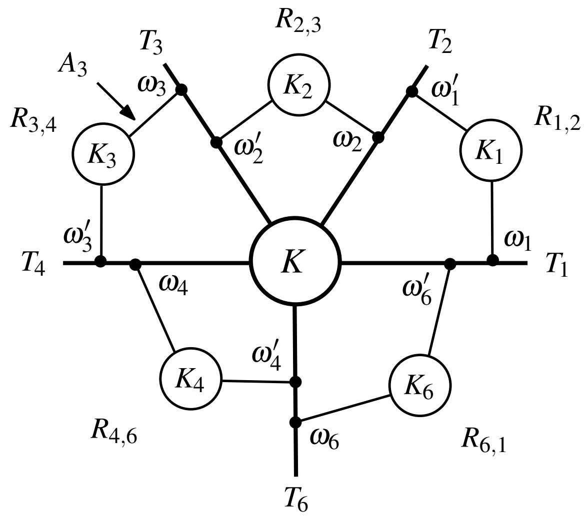

Figure 23: The knot $K = K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6)$.

and non-parallel once-punctured tori, simply by setting $p_5 = 1$ so that T_5 and T_6 become mutually parallel in $R_{4,1}$, and 4 and 6 become consecutive labels.

After setting $p_5 = 1$, a simple strategy to obtain $M = \mathbb{S}^3$ consists in choosing some of the parameters p_i, q_i in such a way that the circle ∂D_4, say, is primitive in $R_{1,4}$, so that $R_{1,4}(\partial D_4)$ is a solid torus and hence $M = R_{1,4}(\partial D_4 \sqcup \partial D_6)$ is a lens space. Choosing the remaining parameters so that the circles $\partial D_4, \partial D_6$ represent an integral homology basis for $R_{1,4}$ finally yields that $M = \mathbb{S}^3$.

We remark that the symmetry between the words of ∂D_4 and ∂D_6 in $\pi_1(R_{1,4})$ discussed in Section 7.4.4 makes irrelevant which of these two circles is chosen to be primitive in $R_{1,4}$, and also that it does not seem possible to implement this strategy using a type 2 Heegaard diagram for $R_{1,4} \cup_\partial R_{4,1}$.

For the rest of this section we will use the notation set up in §7.4. We implement the strategy outlined above by setting the *standard parameters*

$$n = 0, \quad q_1 = \pm 1, \quad p_2 = 2, \quad \delta_3 = \pm 1, \quad q_3 = -(p_3 + \delta_3),$$
$$(p_4, q_4) = (2, 1), \quad p_5 = 1,$$

on top of the generic conditions $p_1, p_3, p_6 \geq 2$ and $\gcd(p_i, q_i) = 1$.

As in §7.4, x, y and x_4, x_6 denote circles dual to the complete disk systems $D_1, D_3 \subset R_{1,4}$ and $D_4, D_6 \subset R_{4,1}$, respectively, so that $\pi_1(R_{1,4}) = \langle x, y \mid - \rangle$ and $\pi_1(R_{4,1}) = \langle x_4, x_6 \mid - \rangle$. Therefore, in $\pi_1(R_{1,4})$, we have

$$\alpha = x^{q_1} y^{p_3} x^{p_1}, \quad \beta = X^{p_1} Y^{p_3} X^{p_1} Y^{q_3}, \quad u = Y^{q_3} X^{p_1} Y^{p_3}, \quad v = y^{p_3} x^{p_1} y^{p_3} x^{q_1},$$

and hence $\partial D_4 = \partial D_4^- = \alpha^2\beta = x^{q_1}y^{p_3}x^{q_1}y^{p_3+\delta_3}$ is primitive in $\pi_1(R_{1,4})$.

From the proof of Lemma 7.11 we have that, in $\pi_1(R_{4,1})$,

$$\omega_1 = (x_4^{p_4}x_6^{p_6})^{p_5-1}x_4^{p_4}x_6^{q_6} = x_4^2x_6^{q_6},$$
$$\omega_3' = (x_6^{p_6}x_4^{p_4})^{p_5-1}x_6^{p_6}x_4^{q_4} = x_4x_6^{p_6},$$

while from Fig. 19(a) we obtain, in $\pi_1(R_{1,4})$,

$$\omega_4 = (x^{p_1}y^{p_3})^{p_2-1}x^{p_1}y^{q_3} = x^{p_1}y^{p_3}x^{p_1}Y^{p_3+\delta_3},$$
$$\omega_6' = (y^{p_3}x^{p_1})^{p_2-1}y^{p_3}x^{q_1} = y^{p_3}x^{p_1}y^{p_3}x^{q_1},$$

relative to base points at the orientation arrows for ω_4 and ω_6' indicated in Figure 17(b). In particular, the circle $\omega_3' \subset T_4$ is primitive in $R_{4,1}$.

By construction we still have that $\omega_4 \subset T_4$ and $\omega_6' \subset T_1$ are power circles in $R_{4,1}$, so T_5 and T_6 are the tori in $R_{4,1}$ induced by the power circles $\omega_4 \subset T_4$ and $\omega_6' \subset T_1$. Since the circle $\omega_3' \subset T_4$ is primitive in $R_{4,1}$, by Lemma 6.8(1)(d) T_5 and T_6 are indeed mutually parallel in $R_{4,1}$ and can be identified with one another, whence by Lemma 6.8(2)(b) we must have $\Delta(\omega_4', \omega_6) = 1$ in $T_5 = T_6$.

The knot $K \subset \partial R_{1,4} \subset M$ now depends on 6 parameters and will be denoted

$$K = K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6) \subset M,$$

with the 5 once-punctured tori $\mathbb{T} = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_6 \subset X_K$ and the core knots K_i of the complementary regions of $\mathbb{T}$ represented by the diagram in Fig. 23, obtained by setting $T_5 = T_6$ in Fig. 12. Homologically, in $R_{1,4}$ we have

$$\partial D_4 = 2\alpha + \beta = 2q_1x + (p_3 - q_3)y = 2q_1x + (2p_3 + \delta_3)y,$$
$$\partial D_6 = p_6u + q_6v = [q_6(p_1 + q_1) - p_1p_6]x + [2p_3q_6 + \delta_3p_6]y,$$

and so

$$M = \mathbb{S}^3 \iff \partial D_4, \partial D_6 \text{ form a basis for the first homology of } R_{1,4}$$
$$\iff \det\begin{bmatrix} 2q_1 & 2p_3 + \delta_3 \\ (p_1 + q_1)q_6 - p_1p_6 & 2p_3q_6 + \delta_3p_6 \end{bmatrix} = Ap_6 + Bq_6 = \varepsilon \in \{\pm 1\},$$

where $A = p_1(2p_3 + \delta_3) + 2\delta_3q_1$ and $B = q_1(2p_3 - \delta_3) - p_1(2p_3 + \delta_3)$.

Lemma 8.2. $\gcd(A, B) = 1$ *for any of the standard values of* p_i, q_i, δ_3*; in particular, there are infinitely many pairs* (p_6, q_6) *with* $p_6 \geq 2$ *such that* $M = \mathbb{S}^3$*, for which* $q_6 > p_6/2 \geq 1$.

Proof. Since $\delta_3, q_1 \in \{\pm 1\}$, we have that $A+B = q_1(2p_3+\delta_3)$ is odd and $A - q_1p_1(A+B) = 2\delta_3 q_1 = \pm 2$, hence that $\gcd(A,B) = 1$. The estimates

$$\begin{aligned} A &= p_1(2p_3+\delta_3) + 2\delta_3 q_1 \geq 2(2p_3-1) - 2 \geq 4 \\ -B &= p_1(2p_3+\delta_3) - q_1(2p_3-\delta_3) \geq p_1(2p_3-1) - (2p_3+1) \\ &= (p_1-1)(2p_3-1) - 2 \geq 1 \end{aligned}$$

show that $q_6 \geq 1$. The relations

$$\begin{aligned} q_6 &= \frac{Ap_6 - \varepsilon}{-B} = \frac{p_6(-B) + p_6(A+B) - \varepsilon}{-B} \\ &= p_6 + \frac{q_1 p_6(2p_3+\delta_3) - \varepsilon}{-B} = p_6 + q_1 \cdot \frac{p_6(2p_3+\delta_3) - \varepsilon q_1}{p_1(2p_3+\delta_3) - q_1(2p_3-\delta_3)} \end{aligned}$$

imply that $q_6 > p_6$ for $q_1 = +1$, while for $q_1 = -1$, since $\varepsilon \leq 1 < 2p_3 - \delta_3$, we have

$$\begin{aligned} 0 < \frac{p_6(2p_3+\delta_3) + \varepsilon}{p_1(2p_3+\delta_3) + (2p_3-\delta_3)} &< \frac{p_6(2p_3+\delta_3) + (2p_3-\delta_3)}{p_1(2p_3+\delta_3) + (2p_3-\delta_3)} \\ &\leq \frac{p_6(2p_3+\delta_3) + (2p_3-\delta_3)}{2(2p_3+\delta_3) + (2p_3-\delta_3)} \leq \frac{p_6}{2} \end{aligned}$$

and hence that $q_6 - \frac{p_6}{2} \geq \frac{p_6}{2} - \frac{p_6(2p_3+\delta_3) + \varepsilon}{p_1(2p_3+\delta_3) + (2p_3-\delta_3)} > 0$. □

Let $\mathscr{K}$ denote the family of all knots $K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6) \subset \mathbb{S}^3$ with standard parameters such that $Ap_6 + Bq_6 = \varepsilon \in \{\pm 1\}$.

Proof of Theorem 2: For each knot $K \in \mathscr{K}$ there is a collection $\mathbb{T} = T_1 \sqcup T_2 \sqcup T_3 \sqcup T_4 \sqcup T_6 \subset X_K$ of 5 mutually disjoint once-punctured tori such that for each i the region $R_{i,i+1}$ is a handlebody and the circles $\omega'_{i-1}, \omega_i \subset T_i$ are power circles in $R_{i-1,i}, R_{i,i+1}$, respectively, with $\Delta(\omega'_{i-1}, \omega_i) = 1$. If there is a circle γ in T_i which is a power in X_K on either side of T_i then, by Lemma 3.1 applied to $R_{i,i}$, γ must be isotopic in T_i to ω'_{i-1} and ω_i, contradicting the fact that $\Delta(\omega'_{i-1}, \omega_i) = 1$. Therefore, by Lemma 8.1 the knot K is hyperbolic and the slope $r = a/b$ of any exceptional surgery on K satisfies the condition $|a| = \Delta(r, J) \leq 1$, so $X_K(r)$ is an integral homology 3-sphere.

Moreover, each pair $(R_{i,i+1}, J)$ is simple of index $p_i \geq 2$, so by Lemma 6.2(4) $X_K(J)$ is the union of Seifert fiber spaces of the form $\mathbb{A}^2(p_1)$, $\mathbb{A}^2(p_2)$, $\mathbb{A}^2(p_3)$, $\mathbb{A}^2(p_4)$, $\mathbb{A}^2(p_6)$, and hence the collection $\widehat{\mathbb{T}}$ produces the JSJ decomposition of

$X_K(J)$. As the manifolds $\mathbb{A}^2(p)$ and $\mathbb{A}^2(q)$ are not homeomorphic for $p \neq q$ (see [11, VI.16]), if $\{p_1, p_3, p_6\} \neq \{p'_1, p'_3, p'_6\}$ then for the knots

$$K = K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6) \in \mathscr{K} \quad \text{and} \quad K' = K^{(1)}(p'_1, q'_1, p'_3, \delta'_3, p'_6, q'_6) \in \mathscr{K}$$

the surgery manifolds $X_K(J)$ and $X_{K'}(J')$ are not homeomorphic, hence K and K' are knots of different types and so by Lemma 8.2 the family of knots $\mathscr{K}$ is infinite. □

The following result establishes a connection between the hyperbolic knots in the family $\mathscr{K}$ and the hyperbolic Eudave-Muñoz knots.

Lemma 8.3. *For each knot $K = K^{(1)}(p_1, q_1, p_3, \delta_3, p_6, q_6) \in \mathscr{K}$ the core knot K_4 of the simple pair $(R_{4,6}, K)$ is a hyperbolic Eudave-Muñoz knot; if $(p_1, q_1) \neq (2,1)$ then K_2 is also a hyperbolic Eudave-Muñoz knot, and otherwise a trivial or cable knot.*

Proof. By construction, the power circles $\omega'_{i-1} \subset T_i \subset R_{i-1,i}$ and $\omega_i \subset T_i \subset R_{i,i+1}$ intersect minimally in one point, hence each region $R_{1,3}, R_{2,4}, R_{3,6}, R_{4,1}$ and $R_{6,2}$ is a handlebody by Lemma 7.2(3).

As $\omega_1 = x_4^2 x_6^{q_6} \in \pi_1(R_{4,1})$ and $q_6 \geq 2$ by Lemma 8.2, ω_1 is a Seifert circle in $R_{4,1}$ and so by Lemma 3.5(1) $R_{4,2}$ is not a handlebody.

Since D_1, D_3 are the compression disks of $\partial R_{1,4} \setminus \omega'_3, \partial R_{1,4} \setminus \omega_1$ in $R_{1,4}$, respectively, the set up in §7.4.1 applies and so by Lemma 7.9 the circle $\omega_4 = x^{p_1} y^{p_3} x^{p_1} Y^{p_3+\delta_3} \in \pi_1(R_{1,4}) = \langle x, y \mid - \rangle$ is represented by the word $\omega_4 = z y^{p_3} z Y^{p_3+\delta_3}$ in $\pi_1(R_{2,4}) = \langle z, y \mid - \rangle$, where $\omega'_1 = z$. Since $p_3 + \delta_3 = p_3 \pm 1 \geq 1$, the word $\omega_4 = z y^{p_3} z Y^{p_3+\delta_3}$ is not primitive in $\langle z, y \mid - \rangle$ by Lemma 7.10 and so $R_{2,6}$ is not a handlebody by Lemma 3.5(1). Therefore, by Lemma 7.1 applied to the collection $T_2, T_4, T_6 \subset X_K$, it follows that K_4 is a hyperbolic Eudave-Muñoz knot.

Since $\omega'_6 = y^{p_3} x^{p_1} y^{p_3} x^{q_1} \in \pi_1(R_{1,4}) = \langle x, y \mid - \rangle$ and $q_1 = \pm 1$, we have by Lemmas 6.8(1)(d) and 7.9 that

$$\begin{aligned} \omega'_6 \text{ is a Seifert circle in } R_{1,4} &\iff \omega'_6 = t x^{p_1} t x^{q_1} \text{ is primitive in } \pi_1(R_{1,3}) = \langle x, t \mid - \rangle \\ &\iff (p_1, q_1) = (2,1). \end{aligned}$$

Thus, by Lemma 3.5(1), $R_{6,3}$ is a handlebody iff $(p_1, q_1) = (2,1)$. Therefore, if $(p_1, q_1) \neq (2,1)$ then $R_{6,3}$ is not a handlebody and so K_2 is a hyperbolic Eudave-Muñoz knot by Lemma 7.1 applied to the collection $T_2, T_3, T_6 \subset X_K$.

For the case $(p_1, q_1) = (2,1)$, since $R_{1,4}$ is a handlebody and the pair $(R_{2,3}, K)$ is simple, by Lemmas 3.5 and 6.4 the circles $\omega'_1 \subset T_2$ and $\omega_3 \subset T_3$ are basic in $R_{2,3}$ and there is an integral slope $s_2 \subset N(K_2) \subset R_{2,3}$ which is coannular in $R_{2,3} \setminus \operatorname{int} N(K_2)$ to a circle $s'_2 \subset \partial R_{2,3} \setminus (\omega'_1 \sqcup \omega_3)$ which intersects each of the power circles $\omega_2 \subset T_2$

and $\omega_2' \subset T_3$ minimally in one point, whence s_2' intersects $K \subset \partial R_{2,3}$ minimally in two points; also, s_2' is a primitive circle in $R_{2,3}$ and the circles ω_1', ω_3 run once around the solid torus $R_{2,3}(s_2')$.

By Lemma 6.3, s_2' can be isotoped in $R_{1,4}$ onto a circle $\widetilde{K}_2$ in $\partial R_{1,4} \setminus (\omega_1 \sqcup \omega_3')$ so that it intersects $K \subset \partial R_{1,4}$ minimally in two points, hence each of the circles ω_4, ω_6' minimally in one point. Thus $\widetilde{K}_2$ must be the circle shown in Figs. 19(a) or (b) (where $p_2 = 2$), modulo some number $m \in \mathbb{Z}$ of Dehn twists along the annulus $A_K \subset \partial R_{1,4}$.

Moreover, by Lemmas 6.3 and 6.4(4) the manifold $R_{1,4}(\widetilde{K}_2)$ is homeomorphic to the union of the solid torus $R_{2,3}(s_2')$ and the companion solid tori of the power circles ω_1', ω_3 in $R_{1,2}, R_{3,4}$, respectively and hence it is a Seifert fiber space of the form $\mathbb{D}^2(p_1, p_4)$, so $\widetilde{K}_2$ is a Seifert circle in $R_{1,4}$.

In the case of the circle $\widetilde{K}_2$ in Fig. 19(a), in $\pi_1(R_{1,4}) = \langle x, y \mid - \rangle$, the word represented by $\widetilde{K}_2$ is of the form

$$w(x^2, y^{p_3}) = y^{p_3} x^2 \Big[X^2 Y^{p_3} x^2 y^{p_3} x^2 y^{p_3} X^2 Y^{p_3} \Big]^m \Big[X^2 y^{p_3} x^2 y^{p_3} x^2 Y^{p_3} X^2 Y^{p_3} \Big]^m,$$

and it is not hard to see that if $m \neq 0$ then the cyclic reduction of the word $w(x, y^{p_3})$ contains both x and X (and y^{p_3} and Y^{p_3}) and hence it is not a primitive word by Lemma 7.10, which by Lemmas 6.8(1)(d) and 7.9 implies that $\widetilde{K}_2$ is not a Seifert circle in $R_{1,4}$, contradicting the above argument. Therefore we must have $m = 0$ and so $\widetilde{K}_2 \subset \partial R_{1,4}$ is isotopic to the circle shown in Fig. 19(a). In the case of the circle $\widetilde{K}_2$ of Fig. 19(b) a similar computation shows that the word $w(x, y^{p_3})$ is not primitive for any $m \in \mathbb{Z}$ and so this case does not arise.

It follows that the circle $\partial D_4 = p_4\alpha + q_4\beta = 2\alpha + \beta \subset \partial R_{1,4} = \partial R_{4,1}$, obtained from Fig. 19(b) with $p_5 = 1$, intersects $\widetilde{K}_2$ minimally in one point and so $\widetilde{K}_2$ is a primitive circle in $R_{4,1}$. The proof of Lemma 3.3(1) now shows that the unique compression disk $E \subset R_{4,1}$ for the surface $\partial R_{4,1} \setminus \widetilde{K}_2$ can be made disjoint from D_4.

Since $\widetilde{K}_2$ is isotopic in $\mathbb{S}^3$ to K_2, we can therefore identify the exterior $X_2 \subset \mathbb{S}^3$ of the knot K_2 with the manifold $R_{1,4}(\partial E)$, so that $\partial D_4 \subset \partial X_2$ is the meridian slope and $\widetilde{K}_2 \subset \partial X_2$ has integral slope.

Now, relative to the point $\widetilde{K}_2 \cap \partial D_4 \subset \partial R_{1,4}$, the words in $\pi_1(R_{1,4}) = \langle x, y \mid - \rangle$ represented by the circles $\widetilde{K}_2$ and ∂D_4 (oriented as in Figs. 19(a),(b)) are

$$\widetilde{K}_2 = y^{p_3} x^2 \quad \text{and} \quad \partial D_4 = y^{p_3} x Y^{q_3} x.$$

If $|q_3| = p_3 + \delta_3 = 1$ then $p_3 = 2$, $\delta_3 = -1$, and $q_3 = -1$, in which case we have that

$$\partial D_4 \cdot (\widetilde{K}_2)^{-1} \cdot \partial D_4 = y^{p_3} x Y^{2q_3} x = (y^2 x)^2$$

and hence $2 \cdot \partial D_4 - \widetilde{K}_2 \subset \partial R_{1,4}$ (written homologically) is a power circle in $R_{1,4}$, while if $|q_3| \geq 2$ then

$$(\widetilde{K}_2)^{-1} \cdot \partial D_4 = (XYx)^{q_3}$$

and hence $\partial D_4 - \widetilde{K}_2 \subset \partial R_{1,4}$ is a power circle in $R_{1,4}$. Therefore in all cases there is a circle $\gamma \subset N(\partial D_4 \cup \widetilde{K}_2) \subset \partial R_{1,4}$ which is a power in $R_{1,4}$ and is disjoint from ∂E, hence the companion annulus and companion solid torus of γ in $R_{1,4}$ lie in $X_2 = R_{1,4}(\partial E)$ and so K_2 is either a trivial or cable knot. □

Remark 8.4. *(1) Other infinite families of hyperbolic knots K in $\mathbb{S}^3$ with a collection $\mathbb{T} \subset X_K$ of 5 once-punctured tori can be obtained using variations of the construction above, for instance, by setting the parameters $n = 0$, $(p_4, q_4) = (1, 0)$, and*

$$(p_1, q_1) = (2, 1), \quad p_2 = p_5 = 2, \quad p_3 \not\equiv 0 \ (mod\ 3), \quad q_3 = \pm 1,$$

along with the conditions $p_3, p_6 \geq 2$ and $\gcd(p_6, q_6) = 1$ on a type 1 Heegaard diagram, in which case the core knot K_5 is always a hyperbolic Eudave-Muñoz knot.

(2) The above process can also be modified to produce examples of hyperbolic knots in $\mathbb{S}^3$ which bound a maximal collection of 4 mutually disjoint and non-parallel once-punctured tori as follows.

On top of the generic conditions $p_1, p_3, p_4, p_6 \geq 2$ and $\gcd(p_i, q_i) = 1$, set the standard values

$$n = 0, \quad p_2 = 1, \quad (p_4, q_4) = (2, 1), \quad p_5 = 1,$$

along with the condition

$$(*) \qquad 2q_1 - p_1 = \delta_1 = \pm 1 \quad or \quad q_3 = \pm 1.$$

Then $A = -(2q_1 - p_1)q_3$ and $B = q_1 q_3 + (2q_1 - p_1)p_3$ are relatively prime integers, and an infinite family of hyperbolic knots $K = K^{(1)}(p_1, q_1, p_3, q_3, p_6, q_6) \subset \mathbb{S}^3$ is produced by the condition $Ap_6 + Bq_6 = \pm 1$, each of which has exterior that contains a family of 4 mutually disjoint and non-parallel once-punctured tori $\mathbb{T} = T_1 \sqcup T_2 \sqcup T_4 \sqcup T_6$ that separate X_K into simple pairs, so that $\widehat{\mathbb{T}}$ produces the JSJ decomposition of $X_K(J)$ consisting of Seifert spaces of the form $\mathbb{A}^2(p_1)$, $\mathbb{A}^2(p_3)$, $\mathbb{A}^2(p_4)$, and $\mathbb{A}^2(p_6)$.

Now, any incompressible torus in $X_K(J)$ can be isotoped away from $\widehat{\mathbb{T}}$ and into the interior of some atoroidal cable space $\mathbb{A}^2(p_k)$, whence it must be isotopic to some component $\widehat{T}_\ell \subset \widehat{\mathbb{T}}$ of $\partial \mathbb{A}^2(p_k)$. So if $\mathbb{T}' = T'_1 \sqcup T'_2 \sqcup T'_3 \sqcup T'_4 \sqcup T'_5 \subset X_K$ is a 5-component maximal family of once-punctured tori then, for some $i \neq j$, $\widehat{T}'_i$ and $\widehat{T}'_j$ must be mutually isotopic, hence parallel, in $X_K(J)$, and hence by Lemma 3.7(4) T'_i and T'_j must be mutually parallel in X_K, which is not the case. Therefore the collection $\mathbb{T}$ is maximal.

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