An Example of Induction: Fibonacci Numbers

Art Duval
University of Texas at El Paso

February 12, 2007

This short document is an example of an induction proof. Our goal is to rigorously prove something we observed experimentally in class, that every fifth Fibonacci number is a multiple of 5.

As usual in mathematics, we have to start by carefully defining the objects we are studying.

Definition. The sequence of **Fibonacci numbers**, $F_0, F_1, F_2, \ldots$, are defined by the following equations:

$$
\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_n &= F_{n-1} + F_{n-2}
\end{aligned}
$$

Theorem 1. *The Fibonacci number F_{5n} is a multiple of 5, for all positive integers n.*

Proof. Proof by induction on n. Since this is a proof by induction, we start with the base case of $n = 1$. That means, in this case, we need to compute $F_{5\times 1} = F_5$. But it is easy to compute (from the definition) that

$$
\begin{aligned}
F_3 &= F_2 + F_1 = 1 + 1 = 2 \\
F_4 &= F_3 + F_2 = 2 + 1 = 3 \\
F_5 &= F_4 + F_3 = 3 + 2 = 5.
\end{aligned}
$$

Thus,

$$F_{5\times 1} = F_5 = 5,$$

which is a multiple of 5.

Now comes the induction step, which is more involved. In the induction step, we assume the statement of our theorem is true for $n = k$, and then prove that is true for $n = k+1$. So assume F_{5k} is a multiple of 5, say

$$F_{5k} = 5x$$

for some integer x. We know that to get anywhere with Fibonacci numbers we need at least two Fibonacci numbers in a row, so here F_{5k+1} is what we'll need. We don't know what F_{5k+1} is, so we can just call it y: Let

$$F_{5k+1} = y$$

for some integer y.

Now compute

$$\begin{aligned} F_{5k+2} &= F_{5k+1} + F_{5k} = y + 5x \\ F_{5k+3} &= F_{5k+2} + F_{5k+1} = (y + 5x) + y = 2y + 5x \\ F_{5k+4} &= F_{5k+3} + F_{5k+2} = (2y + 5x) + (y + 5x) = 3y + 10x \\ F_{5k+5} &= F_{5k+4} + F_{5k+3} = (3y + 10x) + (2y + 5x) = 5y + 15x. \end{aligned}$$

Thus,

$$F_{5(k+1)} = F_{5k+5} = 5y + 15x = 5(y + 3x),$$

which is a multiple of 5, since y and x are integers.

We have shown that if F_{5k} is a multiple of 5, then $F_{5(k+1)}$ is also a multiple of 5. In other words, we have shown that if the statement of our theorem is true for $n = k$, then the statement of our theorem is true for $n = k+1$. That means we have proved the induction step, and thus completed our proof. □