An Example of Induction: Chromatic Polynomials

Art Duval
University of Texas at El Paso

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1 Introduction

This short document is an example of an induction proof. Our goal is to show that the chromatic polynomial really is a polynomial; therefore, from now on, we will call it the chromatic function (to emphasize that we don't know *yet* that it is a polynomial), and denote it by $\chi(G)$.

More formally, then, if G is a graph, we define $\chi(G, x)$ (which we will usually abbreviate as $\chi(G)$) to be the function with variable x whose value is given by the number of ways to properly color the vertices of G with x possible colors.

2 Background material

The following results are the building blocks of our induction. Since you should have seen them before, or should be able to easily prove them, only the barest hints of proof are provided.

Theorem 1. *If p and q are each polynomials (in x), then so are $p + q$ and $p - q$.*

Proof. This is also an induction proof, on the degree of the polynomials. After you undertand the proof the chromatic polynomial, you might want to come back and prove this result carefully, for further practice with induction. Note that once you prove the result for $p + q$, it is easy to prove it for $p - q$, by writing $p - q = p + (-q)$. □

Theorem 2. *If G is the empty graph on k vertices, then $\chi(G) = x^k$.*

Proof. This is an easy counting argument about the chromatic function. $\square$

Recall from class (or the textbook) that, for e and edge of graph G, the deletion of e from G, denoted by $G - e$, is the graph we obtain from G simply by removing edge e from G.

Similarly, the contraction of e from G, denoted by G/e, is the graph we obtain from G by "contracting" the edge e. That is, merge the two vertices u, v that are the endpoints of e into a single vertex, which has an edge to all the other vertices that had an edge with u or v in the original graph G. (This can be defined more formally, but is besides the point of this short note.)

Theorem 3. *If e is an edge of the graph G, then*

$$\chi(G) = \chi(G - e) - \chi(G/e).$$

Proof. We did this in class; it is also explained in the textbook. $\square$

3 The Induction Proof

Now we are ready to use induction to prove our main result.

Theorem 4. *For any graph G, the chromatic function $\chi(G)$ is a polynomial.*

Proof. Proof by induction on the number of edges.

Base case. The fewest number of edges G could have is zero. In this case, by Theorem 2, $\chi(G) = x^k$, for some k, so $\chi(G)$ is a polynomial.

Induction step. Now assume that, if G' is a graph with fewer than n edges, then $\chi(G')$ is a polynomial. [Note: Since we assume the theorem is true for *all* numbers of edges less than n, some people refer to this as "strong induction"; "weak induction" would be if we only assumed the theorem were true for $n - 1$. In practice, most mathematicians do not worry about this distinction.]

Let G be a graph with n edges. By Theorem 3, $\chi(G) = \chi(G-e) - \chi(G/e)$. Now notice that $G - e$ and G/e each have fewer edges than G (in fact, $G - e$ has one fewer edge, namely e, but G/e may be missing more edges, if the endpoints of e had any common neighbors – try some examples!), and so each has fewer than n edges. By the induction hypothesis, then, $\chi(G - e)$ and $\chi(G/e)$ are each polynomials. And then, by Theorem 1, so is $\chi(G)$. $\square$