An Example of Induction: Fibonacci Numbers

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This short document is an example of an induction proof. Our goal is to rigorously prove something we observed experimentally in class, that every fifth Fibonacci number is a multiple of 5.

As usual in mathematics, we have to start by carefully defining the objects we are studying.

Definition. The sequence of **Fibonacci numbers**, $F_0, F_1, F_2, \ldots$, are defined by the following equations:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

We now have to prove one of our early observations, expressing F_{n+5} as a sum of a multiple of 5, and a multiple of F_n.

Lemma 1. *If $n \geq 0$ is an integer, then*

$$F_{n+5} = 5F_{n+1} + 3F_n.$$

Proof. Repeatedly applying the recursion formula for Fibonacci numbers,

$$\begin{aligned} F_{n+5} &= F_{n+4} + F_{n+3} = (F_{n+3} + F_{n+2}) + F_{n+3} \\ &= 2F_{n+3} + F_{n+2} = 2(F_{n+2} + F_{n+1}) + F_{n+2} \\ &= 3F_{n+2} + 2F_{n+1} = 3(F_{n+1} + F_n) + 2F_{n+2} \\ &= 5F_{n+1} + 3F_n. \end{aligned}$$

□

Theorem 2. *The Fibonacci number F_{5k} is a multiple of 5, for all integers $k \geq 1$.*

Proof. Proof by induction on k. Since this is a proof by induction, we start with the base case of $k = 1$. That means, in this case, we need to compute $F_{5\times 1} = F_5$. But, it is easy to compute that $F_5 = 5$, which is a multiple of 5.

Now comes the induction step, which is more involved. In the induction step, we assume the statement of our theorem is true for $k = m$, and then prove that is true for $k = m + 1$. So assume F_{5m} is a multiple of 5, say

$$F_{5m} = 5p$$

for some integer p. We now need to show that $F_{5(m+1)}$ is a multiple of 5. But

$$\begin{aligned} F_{5(m+1)} = F_{5m+5} &= 5F_{5m+1} + 3F_{5m} && \text{by Lemma 1} \\ &= 5F_{5m+1} + (3 \times 5p) && \text{by induction} \\ &= 5(F_{5m+1} + 3p), && \text{by algebra} \end{aligned}$$

which is a multiple of 5 (since F_{5m+1} and p are integers).

We have shown that if F_{5m} is a multiple of 5, then $F_{5(m+1)}$ is also a multiple of 5. In other words, we have shown that if the statement of our theorem is true for $k = m$, then the statement of our theorem is true for $k = m+1$. That means we have proved the induction step, and thus completed our proof. □