An Example of Induction: Fibonacci Numbers

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This short document is an example of an induction proof. Our goal is to rigorously prove something we observed experimentally in class, that every fifth Fibonacci number is a multiple of 5.

As usual in mathematics, we have to start by carefully defining the objects we are studying.

Definition. The sequence of **Fibonacci numbers**, $F_0, F_1, F_2, \ldots$, are defined by the following equations:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n + F_{n+1} = F_{n+2}$$

Theorem 1. *The Fibonacci number F_{5k} is a multiple of 5, for all integers $k \geq 0$.*

Proof. Proof by induction on k. Since this is a proof by induction, we start with the base case of $k = 0$. That means, in this case, we need to compute $F_{5\times 0} = F_0$. But, by definition, $F_0 = 0 = 0 \times 5$, which is a multiple of 5.

Now comes the induction step, which is more involved. In the induction step, we assume the statement of our theorem is true for $k = n$, and then prove that is true for $k = n + 1$. So assume F_{5n} is a multiple of 5, say

$$F_{5n} = 5p$$

for some integer p. We now need to show that $F_{5n+5} = F_{5(n+1)}$ is a multiple of 5. So we repeatedly use the recursive equation that defines Fibonacci numbers, and algebra to compute

$$\begin{aligned}
F_{5n+5} &= F_{5n+3} + F_{5n+4} = F_{5n+3} + (F_{5n+2} + F_{5n+3}) \\
&= F_{5n+2} + 2F_{5n+3} = F_{5n+2} + 2(F_{5n+1} + F_{5n+2}) \\
&= 2F_{5n+1} + 3F_{5n+2} = 2F_{5n+1} + 3(F_{5n} + F_{5n+1}) \\
&= 3F_{5n} + 5F_{5n+1} \\
&= 3(5p) + 5F_{5n+1} && \text{by induction} \\
&= 5(3p + F_{5n+1}), && \text{by algebra}
\end{aligned}$$

which is a multiple of 5 (since F_{5n+1} and p are integers).

We have shown that if F_{5n} is a multiple of 5, then $F_{5(n+1)}$ is also a multiple of 5. In other words, we have shown that if the statement of our theorem is true for $k = n$, then the statement of our theorem is true for $k = n+1$. That means we have proved the induction step, and thus completed our proof. □