Math 4303 | **FUNDAMENTALS OF MATH** | Spring 2017

Dr. Duval | **Closure of algebraic numbers**

Tuesday, April 18

This is an outline to show that the algebraic numbers are closed under addition and multiplication.

1. To show algebraic numbers are closed under addition, let's warm up with an example. Let $\alpha = \sqrt[3]{2}, \beta = \sqrt{3}+1$. We need to find an algebraic equation satisfied by $\alpha+\beta$. We'll show that $\alpha+\beta$ satisfies an algebraic equation of degree 6, without actually constructing the equation.

   (a) We know that α is algebraic because it satisfies the algebraic equation $\alpha^3 - 2 = 0$. We can rewrite this equation in a way that will prove useful: $\alpha^3 = 2$. Now find the algebraic equation satisfied by β [hint: it is a degree 2 polynomial], and use it to find a nice expression for β^2 in terms of β, with only integer coefficients.

   (b) Next write out the first 6 powers of $(\alpha+\beta)$: $(\alpha+\beta), (\alpha+\beta)^2, (\alpha+\beta)^3, \ldots, (\alpha+\beta)^6$. (Pascal's triangle and the binomial theorem will help.) Be sure to "reduce" your answer, using $\alpha^3 = 2$ and the reduction you found in step 1a for β^2, so that the only powers of α in your expansions are 1, α, and α^2, and the only powers of β in your expansions are 1 and β. Of course, you'll also get mixed terms like $\alpha^2\beta$ in your expansion.

   (c) How many different kinds of terms of the form $\alpha^i\beta^j$ (where i and j could be zero) do you get in your expansions in step 1b?

   (d) Explain why, if a polynomial equation (of degree d) shows a number γ is algebraic, then we can think of the polynomial equation as an integer linear combination of $1, \gamma, \gamma^2, \ldots, \gamma^d$ adding up to 0.

   (e) Use step 1c to show why we can find such a linear combination (as described in step 1d) of $1, (\alpha+\beta), (\alpha+\beta)^2, \ldots, (\alpha+\beta)^6$ adding up to 0. Here are two hints to do this, one based on Matrix Algebra, the other on Linear Algebra:

      Matrix Algebra: To make this linear combination add up to 0, it is enough to show that the coefficient on each term $\alpha^i\beta^j$ is 0. Set up this problem as a system of linear equations, one equation for each term $\alpha^i\beta^j$. How many equations do you have? How many variables are you solving for?

      Linear Algebra: Consider the vector space whose basis is all the different kinds of terms you found in step 1c; what is its dimension? Use step 1b to think of $1, (\alpha+\beta), (\alpha+\beta)^2, \ldots, (\alpha+\beta)^6$ as vectors in this vector space; how many of these vectors are there?

2. Now try to replicate this for two arbitrary algebraic numbers α and β, where α is algebraic of degree p, and β is algebraic of degree q.

   (a) Explain why we can write any power of α in terms of $1, \alpha, \alpha^2, \ldots, \alpha^{p-1}$, and any power of β in terms of $1, \beta, \beta^2, \ldots, \beta^{q-1}$.

   (b) When you expand $(\alpha+\beta)^i$ for $i = 1, \ldots, pq$, how many different terms of the form $\alpha^i\beta^j$ (where i and j could be zero) do you get in your expansions?

   (c) Show why we can find a polynomial of degree pq that verifies $\alpha+\beta$ is algebraic.

3. Repeat the reasoning in step 2 to show why, if α is algebraic of degree p, and β is algebraic of degree q, then we can find a polynomial showing that $\alpha\beta$ is algebraic.

4. Final note (for those with some experience in algebra): You've completed the hard part to showing that the algebraic numbers form a field. Can you show why all that's left to do is show that if α is algebraic, then so are $-\alpha$ and $1/\alpha$? To figure out how to do this in general, try some examples:

   Find the roots of $2x^2 + 4x - 3 = 0$, and call them λ and μ. Now find the polynomials that verify $-\lambda$ and $-\mu$ are algebraic (the polynomials that $-\lambda$ and $-\mu$ satisfy). Can you now guess (and then prove) how this works in general?

   Similarly, find the polynomials that verify $1/\lambda$ and $1/\mu$ are algebraic. Can you now guess (and then prove) how this works in general?