Math 4303 | **FUNDAMENTALS OF MATH** | Spring 2017
Dr. Duval | **Patterns in repeating decimals**
Tuesday, April 18

This is an outline to use modular arithmetic to show how some fractions with the same denominator just permute the repeating digits.

1. Some specific prime denominators.

   (a) Here is the first example we saw of this phenomenon (see section 2.1.3, problem 8). The decimal representations of $1/7, 2/7, \ldots$ are:

   $$1/7 = 0.\overline{142857}, \quad 2/7 = 0.\overline{285714}, \ldots$$

   The digits of the period always appear in the same order. Let's see why this is the case.

   In the standard algorithm to compute $1/7$ (it will help if you perform this algorithm carefully right now, writing each step clearly), the first step is to divide 7 into 10, giving a remainder of 3. Because of this remainder of 3, the next step is to divide 7 into $3 \times 10 = 30$. This second step gives a remainder of 2, which means the next step after that is to divide 7 into $2 \times 10 = 20$.

   Make a diagram, where each remainder points to the next remainder:

   $$1 \to 3 \to 2 \to \cdots$$

   Complete this diagram. Why do we start with 1? Use this diagram to explain why the digits of the period in $1/7, 2/7, \ldots$ always appear in the same order.

   (b) The next example we looked at in class was the denominator 13. In this case, instead of getting a single cycle of length 6, we got **two** cycles of length 6. Use the approach above to explain why: The possible remainders now are $1, \ldots, 12$ (why not remainder 0?). Make a diagram where each remainder n points to the the next remainder, *i.e.*, the remainder you get when dividing $10n$ by 13. Use this diagram to describe and explain the pattern of the digits in the period of $1/13, 2/13, \ldots$.

   (c) Repeat this process on (at least) two other prime denominators. At least one of these denominators should have period at least 3.

2. General prime denominator. Let's extend the approach we've taken with 7 and 13 to arbitrary prime denominators. Let p be the prime denominator. (For obvious reasons, let's rule out $p = 2, 5$ for now!)

   (a) Show why we can explain the pattern of the different periods using the diagrams where each remainder $n \pmod p$ points to $10n \pmod p$.

   (b) Prove that in these diagrams, if $a \neq b \pmod p$ then a and b cannot both point to the same c.

   (c) Prove that the diagram must be a disjoint collection of cycles.

(d) Use the following outline to prove that every cycle in the diagram must be the **same length**: Start with the cycle containing 1. If this is the only cycle, then we are done. If not, pick some remainder n that has not shown up yet. Show that the cycle containing n can be arranged so that every entry in the cycle containing n is (mod p) exactly n times the corresponding entry in the cycle containing 1. (It may help to look at the example with $p = 13$, and any other examples you found with more than one cycle.) Use this correspondence to show that the two cycles have the same number of elements.

    If these first two cycles do not use up all the remainders, keep creating new cycles starting with previously unused cycles. Show each new cycle has the same number of elements as the first cycle.

(e) Put this all together to prove that the digits of the periods of the decimal representations of all the fractions with denominator p split into several subsets, such that: the digits within each group are just cyclic permutations of each other; and the size of each subset is exactly equal to the length of the period.

3. Arbitrary fractions. Now we extend these ideas to deal with composite denominators.

    (a) Explain why we only need to look at numerators that are relatively prime to the denominator.

    (b) Explain why we can assume the denominator has no factor of 2 or 5.

    (c) Warm-up: Make the diagram for denominator $m = 21$ (the smallest number that is not prime, and has no factor of 2 or 5).

    (d) Let the denominator be m. Prove that if a is relatively prime to m, and $a \pmod p$ points to $b \pmod p$, then b is also relatively prime to m.

    (e) Repeat the arguments in Step 2 to prove if m has no factors of 2 or 5, then the digits of the periods of the decimal representations of all the fractions in lowest terms with denominator m split into several subsets, such that: the digits within each group are just cyclic permutations of each other; and the size of each subset is exactly equal to the length of the period.