Weighted tree enumeration of cubical complexes

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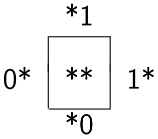
Cubical complexes

Faces of Q_n, n-dimensional cube: $(0, 1, *)$-strings of length n. Dimension is number of *'s.

Vertices: $(0, 1)$-strings of length n

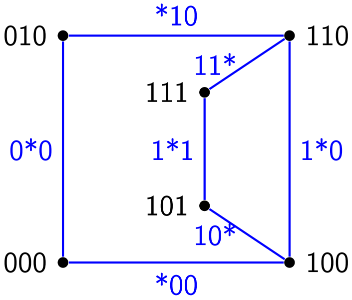
Edge in direction i: single * in position i.

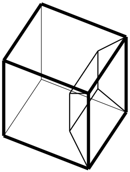
Boundary: faces with one * converted to 0 or 1.

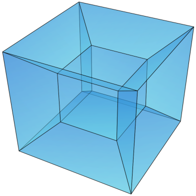


Cubical Complex: Subset of faces of Q_n such that if a face is included, then so is its boundary.

Examples







Spanning trees

Let $\mathcal{Q}$ be a d-dimensional cubical complex.
$\Upsilon \subseteq \mathcal{Q}$ is a cubical spanning tree of $\mathcal{Q}$ when:

0. $\Upsilon_{(d-1)} = \mathcal{Q}_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\mathcal{Q}) - \tilde{\beta}_d(\mathcal{Q})$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
- Works more generally for cellular complexes.
- Inspired by ideas of Kalai on complete simplicial complex.

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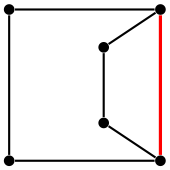

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3×3 w/red edge
15 total

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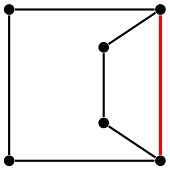

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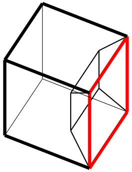

$5 + 5$ w/o red square
5×5 w/red square
35 total

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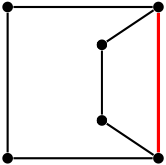

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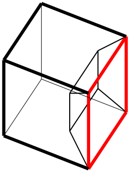

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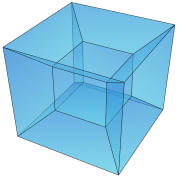
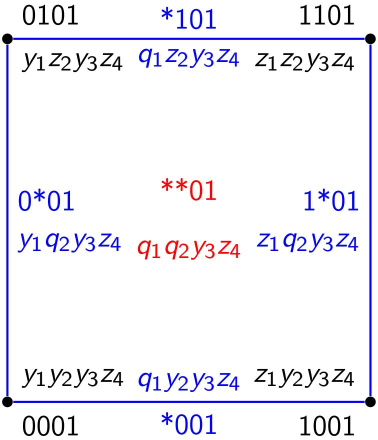


$6^4 \cdot 8^2 = 82944$

Weights



$(y_1q_2y_3)(z_1q_2z_3)(z_1y_2q_3)(q_1y_2y_3)(z_1q_2y_3)$

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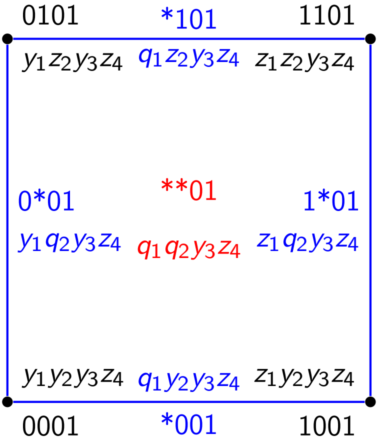
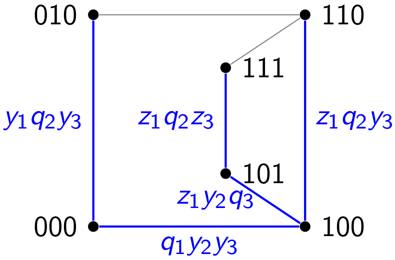




$$(y_1q_2y_3)(z_1q_2z_3)(z_1y_2q_3)(q_1y_2y_3)(z_1q_2y_3)$$

Some motivation from graphs

Let G be a graph with n vertices; let $\partial(G)$ is the oriented boundary matrix (oriented incidence matrix); let

$$L = \partial(G)\partial^T(G)$$

be the Laplacian of G.

Theorem (Kirchoff's Matrix-Tree)

The number of spanning trees of graph G is

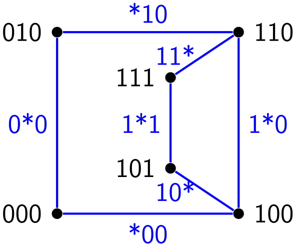
$$(\lambda_1\lambda_2\cdots\lambda_{n-1})/n$$

where $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of L.

Example

$\partial =$

	*00	*10	0*0	1*0	1*1	10*	11*
000	+1	0	+1	0	0	0	0
010	0	+1	-1	0	0	0	0
100	-1	0	0	+1	0	+1	0
101	0	0	0	0	+1	-1	0
110	0	-1	0	-1	0	0	+1
111	0	0	0	0	-1	0	-1



$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

eigenvalues(L): 5,3,3,2,1,0; spanning trees: $(5 \cdot 3 \cdot 3 \cdot 2 \cdot 1)/6 = 15$

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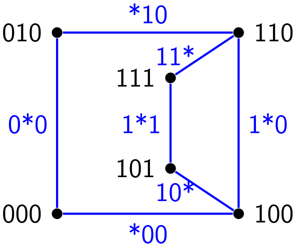

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Weighted Laplacian (arbitrary dimension)

weighted boundary map

$$\hat{\partial} = \begin{array}{c|ccc} & \cdots & \sigma & \cdots \\ \hline \vdots & \ddots & \vdots & \ddots \\ \rho & \cdots & \pm\sqrt{\sigma/\rho} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{array} \qquad \frac{\sigma}{\rho} = \frac{q_i}{x_i} \text{ or } \frac{q_i}{y_i}$$

$\hat{\partial}_k = D_{k-1}^{-1}\partial_k D_k$, where $D_k = \mathrm{diag}(\ldots, \sqrt{\sigma_i^k}, \ldots)$.
(Note that this forms a chain complex: $\hat{\partial}_{k-1}\hat{\partial}_k = 0$.)

weighted Laplacian

$$\hat{L}_{k-1} = \hat{\partial}_k \hat{\partial}_k^T = D_{k-1}^{-1}\partial_k D_k^2 \partial_k^T D_{k-1}^{-1}$$

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Weighted Matrix-Tree Theorem

Recall $\hat{L}_{k-1} = \hat{\partial}_{k-1}\hat{\partial}_{k-1}^T$

Let $\hat{\pi}_k :=$ product of nonzero eigenvalues of $\hat{L}_{k-1}$ [pdet]

Let $X_{(k-1)} :=$ product of all faces of dimension $k-1$

Enumerate spanning trees by
$\hat{\tau}_k := \sum_{\Upsilon \in \mathcal{T}(\mathcal{Q})} |\tilde{H}_{k-1}(\Upsilon)|^2 \, \mathrm{wt}(\Upsilon)$

Theorem (ADM, slight genzn of MMRW)

$$\hat{\pi}_k = \frac{\hat{\tau}_k \hat{\tau}_{k-1}}{|\tilde{H}_{k-1}(\mathcal{Q})| X_{(k-1)}}$$

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Weighted enumeration of spanning trees of $Q_{n,k}$

Conjecture (D-Klivans-M)

$$\hat{\tau}_k(Q_n) = q_{[n]}^{\sum_{i=k-1}^{n-1} \binom{n-1}{i}\binom{i-1}{k-2}} \prod_{\substack{S\subseteq[n] \\ |S|>k}} (u_S y_S z_S)^{\binom{|S|-2}{k-1}}$$

where $u_S = \sum_{i\in S} q_i(\frac{1}{y_i} + \frac{1}{z_i})$

Proof.

Relies on linear algebra tricks from $\hat{\partial}$ making chain complex, and eigenvalues of Q_n (DKM). □

Example

$\hat{\tau}_2(Q_4) = q_{[4]}^7 (u_{123}y_{123}z_{123})^1 \cdots (u_{234}y_{234}z_{234})^1 (u_{[4]}y_{[4]}z_{[4]})^2$

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