Equivalence relations in mathematics, K-16+

Art Duval

Department of Mathematical Sciences
University of Texas at El Paso

MAA Southwestern Sectional Meeting
El Paso Community College
April 19, 2015

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15}$$

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$.

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$.

Questions:

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$.

Questions:

- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$.

Questions:

- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?

One reason fractions are hard

$$\frac{2}{3} + \frac{1}{5} =$$

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$.

Questions:

- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?
- Would we use something else in another situation, or should we always use $\frac{10}{15}$?

Equivalent fractions

Definition: $\frac{a}{b} \sim \frac{c}{d}$ if they reduce to the same fraction ($ad = bc$).

Equivalent fractions

Definition: $\frac{a}{b} \sim \frac{c}{d}$ if they reduce to the same fraction ($ad = bc$).
It's easy to check $\sim$ is an equivalence relation,

Equivalent fractions

Definition: $\frac{a}{b} \sim \frac{c}{d}$ if they reduce to the same fraction ($ad = bc$). It's easy to check $\sim$ is an equivalence relation, so we can partition fractions as follows:

$\frac{a}{b}$ and $\frac{c}{d}$ are in the same part ("equivalence class") if $\frac{a}{b} \sim \frac{c}{d}$.

$\frac{1}{2}$	$\frac{17}{34}$	$\frac{2}{3}$	$\frac{10}{15}$	$\frac{1}{5}$	$\frac{3}{15}$	$\frac{4}{7}$	$\frac{20}{35}$
$\frac{4}{8}$	$\frac{6}{12}$	$\frac{4}{6}$	$\frac{14}{21}$	$\frac{10}{50}$	$\frac{8}{40}$	$\frac{40}{70}$	$\frac{16}{28}$
$\frac{10}{20}$	$\frac{7}{14}$	$\frac{20}{20}$	$\frac{8}{12}$	$\frac{2}{10}$	$\frac{7}{35}$	$\frac{8}{14}$	$\frac{36}{63}$

Adding fractions (revisited)

if	$\frac{a}{b} \sim \frac{c}{d}$
and	$\frac{e}{f} \sim \frac{g}{h}$

Adding fractions (revisited)

$$\begin{array}{rc}
\text{if} & \frac{a}{b} \sim \frac{c}{d} \\
\text{and} & \frac{e}{f} \sim \frac{g}{h} \\
\hline
\text{then} & \frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{h}
\end{array}$$

Adding fractions (revisited)

$$\begin{array}{rr} \text{if} & \frac{a}{b} \sim \frac{c}{d} \\ \text{and} & \frac{e}{f} \sim \frac{g}{h} \\ \hline \text{then} & \frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{h} \end{array}$$

So, really we should say

$$\left[\frac{2}{3}\right] + \left[\frac{1}{5}\right] = \left[\frac{13}{15}\right],$$

because anything equivalent to $\frac{2}{3}$ plus anything equivalent to $\frac{1}{5}$ "equals" something equivalent to $\frac{13}{15}$.

Adding fractions (revisited)

$$\begin{array}{rc} \text{if} & \frac{a}{b} \sim \frac{c}{d} \\ \text{and} & \frac{e}{f} \sim \frac{g}{h} \\ \hline \text{then} & \frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{h} \end{array}$$

So, really we should say

$$\left[\frac{2}{3}\right] + \left[\frac{1}{5}\right] = \left[\frac{13}{15}\right],$$

because anything equivalent to $\frac{2}{3}$ plus anything equivalent to $\frac{1}{5}$ "equals" something equivalent to $\frac{13}{15}$.

- But it's hard to compute unless we pick the right representative.

Adding fractions (revisited)

$$\begin{array}{rrcl} \text{if} & \frac{a}{b} & \sim & \frac{c}{d} \\ \text{and} & \frac{e}{f} & \sim & \frac{g}{h} \\ \hline \text{then} & \frac{a}{b}+\frac{e}{f} & \sim & \frac{c}{d}+\frac{g}{h} \end{array}$$

So, really we should say

$$\left[\frac{2}{3}\right] + \left[\frac{1}{5}\right] = \left[\frac{13}{15}\right],$$

because anything equivalent to $\frac{2}{3}$ plus anything equivalent to $\frac{1}{5}$ "equals" something equivalent to $\frac{13}{15}$.

- But it's hard to compute unless we pick the right representative.
- In other settings, we stick to the fraction in lowest terms, a distinguished representative.

Similarity, congruence, etc.

Some equivalence relations from geometry:

Similarity, congruence, etc.

Some equivalence relations from geometry:

- Similarity
  - same “shape”, possibly different size
  - can get via dilation, reflection, rotation, translation

Similarity, congruence, etc.

Some equivalence relations from geometry:

- Similarity
  - same “shape”, possibly different size
  - can get via dilation, reflection, rotation, translation
- Congruence
  - same “shape”, size
  - can get via reflection, rotation, translation

Similarity, congruence, etc.

Some equivalence relations from geometry:

- Similarity
  - same "shape", possibly different size
  - can get via dilation, reflection, rotation, translation
- Congruence
  - same "shape", size
  - can get via reflection, rotation, translation
- Same shape, size, chirality
  - can get via rotation, translation

Similarity, congruence, etc.

Some equivalence relations from geometry:

- Similarity
  - same “shape”, possibly different size
  - can get via dilation, reflection, rotation, translation
- Congruence
  - same “shape”, size
  - can get via reflection, rotation, translation
- Same shape, size, chirality
  - can get via rotation, translation
- Same shape, size, chirality, orientation
  - can get via translation

Similarity, congruence, etc.

Some equivalence relations from geometry:

- Similarity
  - same "shape", possibly different size
  - can get via dilation, reflection, rotation, translation
- Congruence
  - same "shape", size
  - can get via reflection, rotation, translation
- Same shape, size, chirality
  - can get via rotation, translation
- Same shape, size, chirality, orientation
  - can get via translation
- Same shape, size, chirality, orientation, position
  - equality

Finer partitions

Finer partitions

- As we go down that ladder, we refine the partition, by splitting each part into more parts.

Finer partitions

- As we go down that ladder, we refine the partition, by splitting each part into more parts.
- Different situations call for different interpretations of when two shapes are "the same".

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.
- At my vending machine, dollar good, coins bad.

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.
- At my vending machine, dollar good, coins bad.
- At parking meters, quarters good, everything else bad.

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.
- At my vending machine, dollar good, coins bad.
- At parking meters, quarters good, everything else bad.
- Everywhere, pennies bad.

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids and no knife.

Where else do we see this?

Where else do we see this?

Glad you asked

Regrouping

To do multidigit addition and subtraction,

$$436 = 400 + 30 + 6 = 400 + 20 + 16 = 300 + 130 + 6 = \cdots$$

- Different representations are better or worse for different addition and subtraction problems.
- Using base-10 blocks, these all make different (but "equivalent") pictures.

"Unique" factorization

Completely factor 60, as

$$2 \times 2 \times 3 \times 5 = 2 \times 3 \times 2 \times 5 = 5 \times 2 \times 2 \times 3 = \cdots$$

- Natural to say these are all the "same"; once we do, we get unique factorization into primes.

"Unique" factorization

Completely factor 60, as

$$2 \times 2 \times 3 \times 5 = 2 \times 3 \times 2 \times 5 = 5 \times 2 \times 2 \times 3 = \cdots$$

- Natural to say these are all the "same"; once we do, we get unique factorization into primes.
- Distinguished representative is usually to arrange primes from smallest to largest.
- In context of factorization, 6×10 and 4×15 are different, even though usually $6 \times 10 = 4 \times 15$.

0.999 . . .

$$0.999\ldots = 1$$

0.999 . . .

$$0.999\ldots = 1$$

right?

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process that gets arbitrarily close to 1

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's close enough for everything we do.

$0.999\ldots$

$$0.999\ldots = 1$$

right?

- $0.999\ldots$ isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's close enough for everything we do.
- And allowing it (and all its infinite process buddies) allows us to say things like $\sqrt{2}$ and e are numbers, on the number line.

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's good enough for everything we do.

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's good enough for everything we do.
- But it is not so obvious when expressions are equivalent.

Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's good enough for everything we do.
- But it is not so obvious when expressions are equivalent.
- There are many different ideas of "distinguished representative".

Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$,

Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$, right?

Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$, right?

- The two equations have the same solution set for x.

Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$, right?

- The two equations have the same solution set for x.
- Having the same solution set for [all relevant variables] is an equivalence relation.

Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$, right?

- The two equations have the same solution set for x.
- Having the same solution set for [all relevant variables] is an equivalence relation.
- The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$,

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

- With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

- With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).
- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

- With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).
- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

- With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).
- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
- Usually, the distinguished representative (ordered list) to represent a combination (unordered list) is to put the items "in order"; for instance: $\{1, 10, 17, 23, 42, 54\}$.

Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.

Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.
- But translating that arrow does not change the vector.

Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.
- But translating that arrow does not change the vector.
- So we can think of a vector as an equivalence class of arrows; two arrows are equivalent if they have the same direction and magnitude.

Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.
- But translating that arrow does not change the vector.
- So we can think of a vector as an equivalence class of arrows; two arrows are equivalent if they have the same direction and magnitude.
- Distinguished representative is often to start at the origin. But to see how to add two vectors, we should move the starting point of the second one.

Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.
- But translating that arrow does not change the vector.
- So we can think of a vector as an equivalence class of arrows; two arrows are equivalent if they have the same direction and magnitude.
- Distinguished representative is often to start at the origin. But to see how to add two vectors, we should move the starting point of the second one.
- This equivalence relation respects vector addition and scalar multiplication.

Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by m.
- Example: Even and odd ($m = 2$).

Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by m.
- Example: Even and odd ($m = 2$).
- This equivalence relation respects addition and multiplication.
- Example: Last digit arithmetic ($m = 10$).

Anti-differentiation

Solve

$$f'(x) = 3x^2$$

- "Answer" is $x^3 + C$.
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a constant.
- This equivalence relation respects addition, multiplication by a constant, which is why those are easy to deal with in anti-differentiation.

Linear Differential equations

Solve

$$y''' - 5y'' + y' - y = 3x^2$$

- Solutions of the form

$$y = y_0 + y_p$$

  where y_0 is the general solution to the homogeneous equation, and y_p is a particular solution.
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a solution of the homogeneous equation.

Linear Differential equations

Solve

$$y''' - 5y'' + y' - y = 3x^2$$

- Solutions of the form

$$y = y_0 + y_p$$

  where y_0 is the general solution to the homogeneous equation, and y_p is a particular solution.
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a solution of the homogeneous equation.

Similarly for the matrix equation

$$Mx = b.$$

Gaussian elimination in matrices

- Consists of a series of elementary row operations that do not change the solution set.
- So at the end, we have a nicer representative of the same equivalence class (of systems with the same solution).

Cardinality

What is the cardinality of a set?

- It's not defined as a function, *per se*
- We just say when two sets have the same cardinality.
- That's an equivalence relation, not a function.
- There are some distinguished representatives: $0; 1; 2; \ldots; \mathbb{N}; \mathbb{R}$.

Why do some equivalence relations respect addition?

What we really need is to make sure that [0] acts like the additive identity:

$$[0] + [0] = [0].$$

Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

Why do some equivalence relations respect addition?

What we really need is to make sure that [0] acts like the additive identity:

$$[0] + [0] = [0].$$

Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

If these hold, then it's easy to check that that the equivalence relation respects addition.

Why do some equivalence relations respect addition?

What we really need is to make sure that [0] acts like the additive identity:

$$[0] + [0] = [0].$$

Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

If these hold, then it's easy to check that that the equivalence relation respects addition.

Similarly, the nonabelian case gives rise to normal subgroups.

Why do some equivalence relations respect multiplication?

What we really need is to make sure that [0] acts like the multiplicative "killer":

$$[0] \times [x] = [0]$$

for all $[x]$.
Along with the subgroup condition (for addition), this is just the definition of ideal.

Why do some equivalence relations respect multiplication?

What we really need is to make sure that [0] acts like the multiplicative "killer":

$$[0] \times [x] = [0]$$

for all $[x]$.
Along with the subgroup condition (for addition), this is just the definition of ideal.

If these hold, then it's easy to check that the equivalence relation respects multiplication.