Metric polyhedral complexes: a very preliminary report

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Metric graphs

In this section, we are following Baker-Faber '11.

Definition (metric graph)

Graph G with metric structure (function assigning positive length to each edge).

Remark

We also have to make sure everything plays well with refinement (insert a degree 2 vertex in the middle of an edge, preserving the total length of the edge).
To get to a continuous version, take direct limits of all of the relevant computations, under repeated refinement.

Picard group

Let G be a graph.

Definition (divisor)

Div $X = C_0(G, \mathbb{Z})$. Degree is $\deg(\sum_v a_v[v]) = \sum a_v$.

Definition (principal divisor)

A rational function on G is continuous $\mathbb{R}$-valued, linear on each edge with integer slope; its principal divisor is

$$\operatorname{div}(f) = \sum_v \left(\sum_{e \supset v} \operatorname{slope}(f, v, e) \right) [v] \ \in \ \operatorname{Div} X$$

where $\operatorname{slope}(f, v, e)$ is outgoing slope of f from v to e.

Definition (Picard group)

Principal divisors have degree 0, so we can define

$$\operatorname{Pic}^0(G) = \operatorname{Div}^0(G) / \operatorname{Prin}(G)$$

Jacobian group

Definition (boundary and coboundary maps)

- $\partial\colon C_1(G;\mathbb{R}) \to C_0(G;\mathbb{R})$ as usual
- $de = \partial e / \operatorname{vol}(e)$
- $d^T\colon C^0(G) \to C^1(G)$ by
  $(d^T f)(e) = f(\partial e)/\operatorname{vol}(e) = \operatorname{slope}(f, v, e)$.

Definition (harmonic form)

$\Omega(G) = \{g \in C^1\colon gVf^T = 0 \quad \forall f \in \operatorname{im} d^T\}$

Definition (Jacobian group)

$\Omega(G)^\sharp = \mathbb{Z}\langle\{L_e \in \Omega(G)^*\}\rangle$, where L_e is the volume-weighted functional corresponding to edge e. (Can be interpreted as integration.) $J(G) = \Omega(G)^\sharp / H_1(G;\mathbb{Z})$.

Theorem

$\operatorname{Pic}^0(G) \cong J(G)$.

Example

Digon with edges α, β, with volumes $q, r = 1 - q$ respectively, so

$$\partial = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad d = \begin{bmatrix} 1/q & -1/r \\ -1/q & 1/r \end{bmatrix}$$

$$\begin{aligned} \Omega_1 &= \left\{ g = [g_1, g_2] \colon gVf^T \equiv g_1 q f_1 + g_2 r f_2 = 0 \quad \forall f \in \text{rowspace}\, d \right\} \\ &= \{[g_1, g_2] \colon g_1 q/q - g_2 r/r = 0\} \\ &= \mathbb{R}\langle [1,1] \rangle = \mathbb{R}\langle \alpha^* + \beta^* \rangle \end{aligned}$$

$C_1(G, \mathbb{Z}) = \mathbb{Z}\langle q\alpha, r\beta \rangle$, and $q\alpha + r\beta$ is a representative for the generator of $H_1(G, \mathbb{Z}) \cong \mathbb{Z}$. Therefore

$$J^1 = (q\mathbb{Z} + r\mathbb{Z})/\mathbb{Z} \cong \begin{cases} \mathbb{Z} & \text{if } r \in \mathbb{Q}, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } r = m/n, \quad m, n \in \mathbb{N}, \quad \gcd(m, n) = 1. \end{cases}$$

Cell complexes

How much of this extends to higher dimensions? First review higher dimensional critical groups [DKM '15]

Definition (cell complex)

A cell complex X consists of cells (homeomorphic copies of $\mathbb{R}^k$ for various k) together with attaching maps

$$\partial_k \colon C_k(X) \to C_{k-1}(X)$$

Remark

Like simplicial complexes and boundary maps, except that:

- cells don't have to be simplicial; and
- attaching maps can wrap around the boundary more than once.

Critical and cocritical groups...

Let X be d-dimensional cell complex

Definition (critical group)

$$K(X) := \mathbf{T}(\ker \partial_{d-1} / \operatorname{im} \partial_d \partial_d^T) = \mathbf{T}(\operatorname{coker} \partial_d \partial_d^T)$$

where $\mathbf{T}$ denotes the torsion summand.

Definition (cocritical group)

$$K^*(X) := C_{d+1}(Y; \mathbb{Z}) / \operatorname{im} \partial_{d+1}^T \partial_{d+1} = \operatorname{coker} \partial_{d+1}^T \partial_{d+1}$$

where Y is the acyclization of X: fill in just enough d-dimensional cycles of X with $(d+1)$-dimensional faces to remove all d-dimensional homology.

...which are not quite isomorphic

Theorem (DKM '15)

If X is a d-dimensional cell complex, then

$$0 \to \mathit{cutflow} \to K(X) \to \mathbf{T}(\tilde{H}_{d-1}(X;\mathbb{Z})) \to 0$$
$$0 \to \mathbf{T}(\tilde{H}_{d-1}(X;\mathbb{Z})) \to \mathit{cutflow} \to K^*(X) \to 0$$

where cutflow is some group that intermediates between critical and cocritical groups.

Corollary

When there is no torsion (e.g., graphs), then

$$K(X) \cong \mathit{cutflow} \cong K^*(X)$$

Metric polyhedral complexes

In this last section, we attempt to marry metric graphs to higher-dimensional work.

Definition (metric polyhedral complex)

Polyhedral cell complex X (dimension d) with metric structure (function assigning positive volume to each face).

Remark

Again we have to make sure everything plays well with refinements (pretty sure) and direct limits (have not verified yet).

Picard group

Let X be a d-dimensional metric polyhedral complex.

Definition (divisor)

$\operatorname{Div} X = C_{d-1}(X, \mathbb{Z})$. Degree is $\deg(\sum_\tau a_\tau[\tau]) = \sum a_\tau \operatorname{vol}(\tau)$.

Definition (principal divisor)

A rational function on X is continuous $\mathbb{R}$-valued, linear on each face with integer coefficients; its principal divisor is

$$\operatorname{div}(f) = \sum_\tau \left(\sum_{\sigma \supset \tau} \operatorname{slope}(f, \tau, \sigma) \right) [\tau] \;\in\; \operatorname{Div} X$$

where $\operatorname{slope}(f, \tau, \sigma)$ is outgoing slope of f from ridge τ to facet σ.

Definition (Picard group)

Principal divisors have degree 0, so we can define

$$\operatorname{Pic}^0(X) = \operatorname{Div}^0(X) / \operatorname{Prin}(X)$$

Jacobian group

Definition (boundary and coboundary maps)

- $\partial_k \colon C_k(X) \to C_{k-1}(X)$ as usual
- $V_k = \mathrm{diag}(\mathrm{vol}(\alpha) \colon \alpha \in X_k)$
- $d_k = V_{k-1}\partial_k V_k^{-1}$

Definition (harmonic k-forms)

$\Omega_k(X) = \{g \in C^k \colon gVf^T = 0 \quad \forall f \in \operatorname{im} d_k \oplus d_{k+1}^T\}$

Definition (Jacobian)

$\Omega(X)^\sharp = \mathbb{Z}\langle\{L_\sigma \in \Omega(X)^*\}\rangle$, where L_σ is the volume-weighted functional corresponding to facet σ. $J^k(X) = \Omega_k(X)^\sharp / H_k(X;\mathbb{Z})$.

Question

How is Picard group related to Jacobian group in higher dimensions?

Example

2-sphere with two 0-cells; oppositely oriented 1-cells of lengths q, r; oppositely oriented 2-cells A, B of areas a, b with $a + b = 1$.

$$C_0 = \mathbb{R}^2 \xleftarrow[\begin{bmatrix} 1/q & -1/r \\ -1/q & 1/r \end{bmatrix}]{d_1} C_1 = \mathbb{R}^2 \xleftarrow[\begin{bmatrix} q/a & -q/b \\ r/a & -r/b \end{bmatrix}]{d_2} C_2 = \mathbb{R}^2.$$

$$\begin{aligned} \Omega_2 &= \left\{ g = [g_1, g_2] \colon gVf^T \equiv g_1 a f_1 + g_2 b f_2 = 0 \quad \forall f = [f_1, f_2] \in \text{rowspace}\, d_2 \right\} \\ &= \{[g_1, g_2] \colon g_1 aq/a - g_2 bq/b = 0\} \\ &= \mathbb{R}\left\langle [1,1] \right\rangle = \mathbb{R}\left\langle A^* + B^* \right\rangle . \end{aligned}$$

$C_2(X, \mathbb{Z}) = \mathbb{Z}\langle aA, bB \rangle$, and $aA + bB$ is a representative for the generator of $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$. As in earlier example, if $a = m/n$ is rational with $\gcd(m, n) = 1$ then J^2 is cyclic of order n, while if a is irrational then $J^2 \cong \mathbb{Z}$.