Enumerating simplicial spanning trees of shifted and color-shifted complexes, using simplicial effective resistance

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Outline

- Enumerating spanning trees of graphs
  - Complete graphs
  - Ferrers graphs
  - Using electrical network theory

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  - Complete graphs
  - Ferrers graphs
  - Using electrical network theory
- Simplicial complexes
  - Simplicial electrical network theory
  - Simplicial spanning trees
  - Special families of simplicial complexes
    - Color-shifted complexes (prove conjecture)
    - Shifted complexes (reprove old result)
    - What else might work?

Spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

$T \subseteq E(G)$ is a **spanning tree** of G when:

0. spanning: T contains all vertices;
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

Counting spanning trees

Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \mathrm{wt}\, T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\mathrm{wt}\, T = \prod_{e \in T} \mathrm{wt}\, e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

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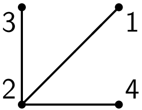
Example (K_4)

- 4 trees like: $T =$ 

$\mathrm{wt}\, T = (x_1 x_2 x_3 x_4) x_2^2$

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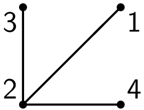
- 4 trees like: $T =$ 

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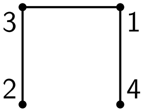
- 12 trees like: $T =$ 

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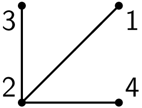
- 4 trees like: $T =$ 

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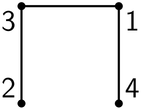
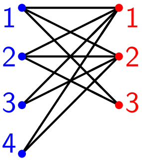
- 12 trees like: $T =$ 

$\mathrm{wt}\, T = (x_1x_2x_3x_4)x_1x_3$
- Total is $(x_1x_2x_3x_4)(x_1 + x_2 + x_3 + x_4)^2$.

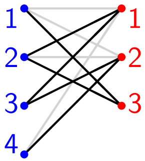
Ferrers graphs (Ehrenborg-van Willigenburg '04)

Example ($\langle 42, 23 \rangle$)

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		



Spanning trees of Ferrers graphs

- $T =$ 

$\mathrm{wt}\, T = (1234)(123)23123$

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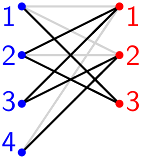
- $T =$ 

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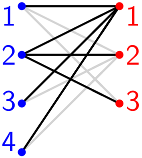
- $T =$ 

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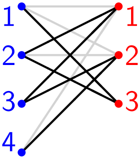
- $T =$ 

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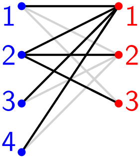
- $T =$ 

wt $T = (1234)(123)2^2 1^3$
- Total is $(1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$

Theorem

$$\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\ \hline
1 & 11 & 21 & 31 & 41 \\ \hline
2 & 12 & 22 & 32 & 42 \\ \hline
3 & 13 & 23 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3}
\end{array}$$

Total is (1234)(123)

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Theorem (Ehrenborg-van Willigenburg)

This works in general

Proof – by electrical network theory!

- Set $I_{ij} = 1$
- Set $R_{pq} = (pq)^{-1}$
- Find remaining currents so they satisfy Kirchhoff's Laws
- Compute V_{ij}, which is effective resistance since $I_{ij} = 1$

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Theorem (Thomassen '90)

$$V_{ij} = \frac{\textit{spanning trees with } ij}{\textit{spanning trees without } ij}$$

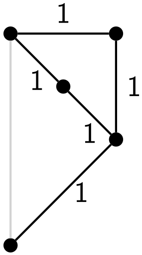
From this, we can easily get

$$\frac{\text{spanning trees of (graph with } ij\text{)}}{\text{spanning trees of (graph without } ij\text{)}}$$

Now apply induction

Example (Unweighted)

Example ($K_{3,2} = \langle 32 \rangle$)



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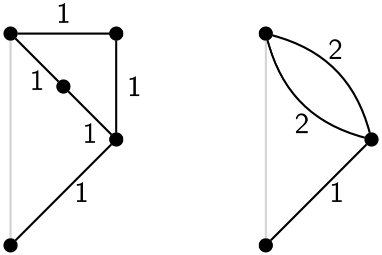

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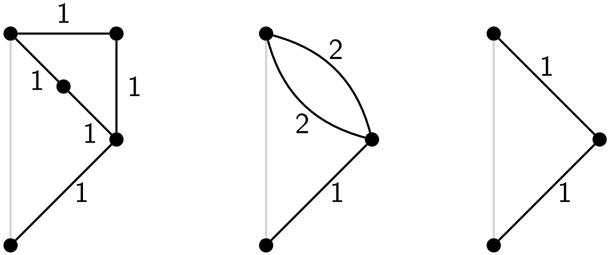

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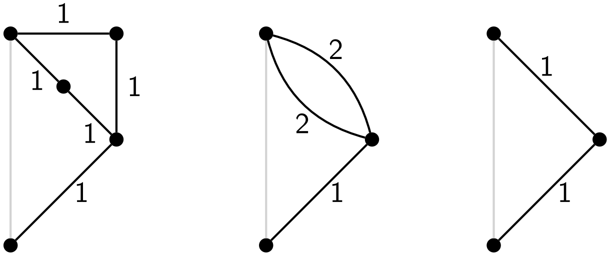

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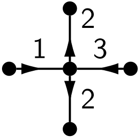
$$\frac{\text{trees with edge}}{\text{trees without edge}} = \frac{8}{4} = 2$$

Kirchhoff's Laws

Start with a simple graph. Each edge has a positive resistance R, directed current I, and directed voltage drop V

Kirchhoff's Laws



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Current
- Sum of currents at a vertex is 0
- $\ker \partial_1$
- spanned by directed cycles

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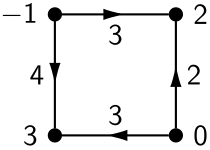

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- there is a potential function ϕ such that $\phi(\partial e) = V_e$.

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Can "solve" circuits by minimizing energy (RI^2 on each edge)

Simplicial networks (Catanzaro-Chernyak-Klein '15)

Start with d-dimensional simplicial complex. Each facet has a positive resistance R, oriented current I, oriented voltage V

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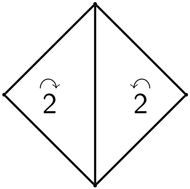

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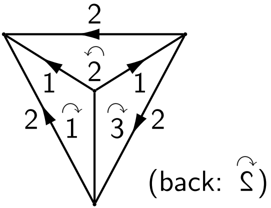

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We still have energy minimization.

Simplicial effective resistance

Let σ be a facet of simplicial complex X

- Set $I_\sigma = 1$
- Set $R_\tau = (x_\tau)^{-1}$ for all other facets τ.
- Assume remaining currents satisfy simplicial network laws
- Compute V_σ, which is effective resistance since $I_\sigma = 1$.

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Theorem (Kook-Lee '18)

$$V_\sigma = \frac{\hat{k}_d(X)_\sigma}{\hat{k}_d(X - \sigma)}$$

where $\hat{k}_d$ is a torsion-weighted simplicial tree count, and $\hat{k}_d(X)_\sigma$ means restricted to trees containing σ.

Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete d-dimensional simplicial complex on n vertices. $T \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $T_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(T; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(T; \mathbb{Z}) = 0$ ("acyclic");
3. $|T| = \binom{n-1}{d}$ ("count").

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- When $d = 1$, coincides with usual definition.

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Example

$n = 5, d = 2 : T = \{123, 124, 125, 134, 135, 245\}$

Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{T \in \mathcal{T}(K_n^d)} \qquad\qquad = n^{\binom{n-2}{d}}$$

Counting simplicial spanning trees of K_n^d

Theorem [Kalai '83]

$$k(K_n^d) = \sum_{T \in \mathcal{T}(K_n^d)} |\tilde{H}_{d-1}(T)|^2 = n^{\binom{n-2}{d}}$$

Weighted simplicial spanning trees of K_n^d

As before,

$$\text{wt}\, T = \prod_{F \in \Upsilon} \text{wt}\, F = \prod_{F \in T} \Big(\prod_{v \in F} x_v \Big)$$

Example

$$\begin{aligned} T &= \{123, 124, 125, 134, 135, 245\} \\ \text{wt}\, T &= x_1^5 x_2^4 x_3^3 x_4^3 x_5^3 \end{aligned}$$

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Theorem (Kalai, '83)

$$\hat{k}(K_n^d) := \sum_{T \in \mathcal{T}(K_n^d)} |\tilde{H}_{d-1}(T)|^2 (\text{wt } T)$$
$$= (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}$$

More generally (D.-Klivans-Martin '09)

Let X be a d-dimensional simplicial complex.
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3. $f_d(T) = f_d(X) - \tilde{\beta}_d(X) + \tilde{\beta}_{d-1}(X)$ ("count").

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$$k_d(X) = \sum_{T \in \mathcal{T}(X)} |\tilde{H}_{d-1}(T,\mathbb{Z})|^2$$

$$\hat{k}_d(X) = \sum_{T \in \mathcal{T}(X)} |\tilde{H}_{d-1}(T,\mathbb{Z})|^2 \operatorname{wt} T$$

Enumeration results

- Skeleta of complete complexes (Kalai '83)
- Complete colorful complexes (Adin '92)
- Shifted complexes (D.-Klivans-Martin, '09)
- Cubical complexes (D.-Klivans-Martin, '11)
- Matroid complexes (Kook-Lee, '16)
- Weighted enumeration of complete colorful and cubical complexes (Aalipour-D.-Kook-Lee-Martin, '18)

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But all rely on Matrix-Tree Theorem, and computing eigenvalues. For Ferrers graphs, even the unweighted eigenvalues are not integers.

Color-shifted complexes

Definition (Babson-Novik, '06)

A color-shifted complex is a simplicial complex with:

- vertex set $V_1 \dot\cup \ldots \dot\cup V_r$ (V_i is set of vertices of color i);
- every facet contains one vertex of each color; and
- if $v < w$ are vertices of the same color, then you can always replace w by v.

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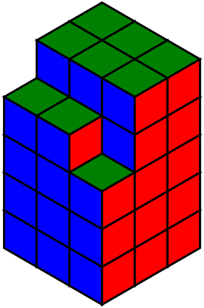
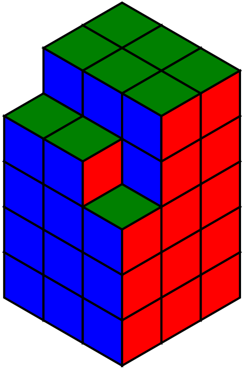
Example

$\langle 235, 324, 333 \rangle$



Enumeration: $\hat{\tau}(\langle 235, 324, 333\rangle)$

$$
\begin{aligned}
&(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4)\\
&\quad \times (1+2+3)^5 (1+2)^3 \; (1+2+3)^6 (1+2)\\
&\quad \times (1+\cdots+5)^2 (1+2+3+4)(1+2+3)
\end{aligned}
$$



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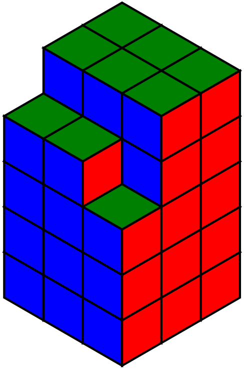

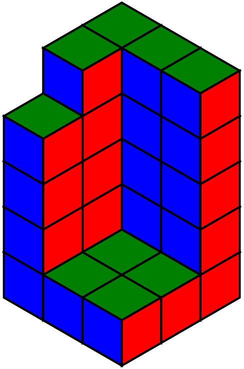

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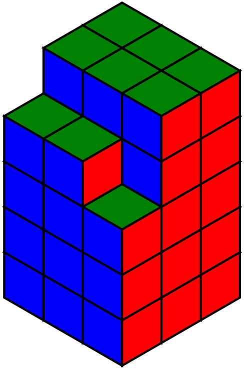
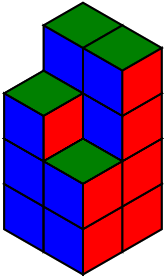
$$\times\ (1+2+3)^5(1+2)^3$$

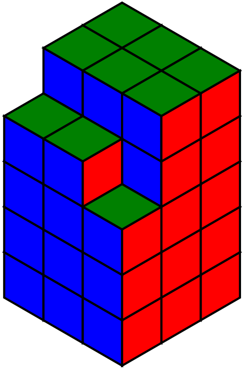


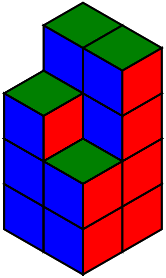


Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$\times$

$$(1+2+3)^6(1+2)$$





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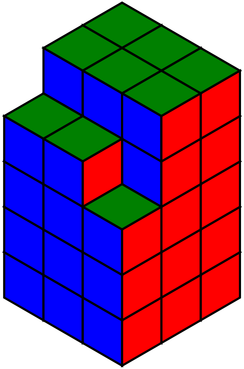
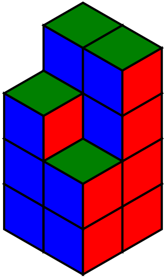
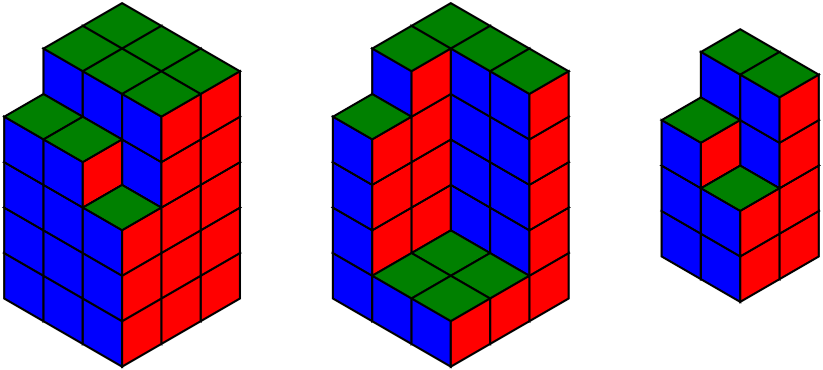
$$\times\, (1 + \cdots + 5)^2 (1 + 2 + 3 + 4)(1 + 2 + 3)$$





Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$
\begin{aligned}
&(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4) \\
&\quad \times (1+2+3)^5 (1+2)^3 \ (1+2+3)^6 (1+2) \\
&\quad \times (1+\cdots+5)^2 (1+2+3+4)(1+2+3)
\end{aligned}
$$



Proof

$$(1^7 2^7 3^6)(1^7 2^7 3^7)(1^5 2^5 3^5 4^5 5^4)$$
$$\times (1+2+3)^5 (1+2)^3 \; (1+2+3)^6 (1+2)$$
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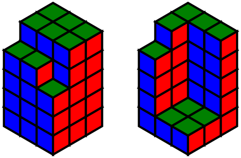

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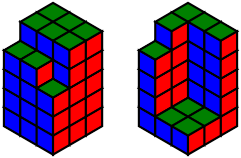

Conjectured by Aalipour-AD (long matrix manipulation pf. $r = 3$)

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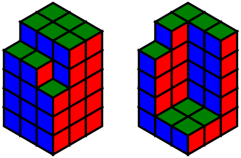

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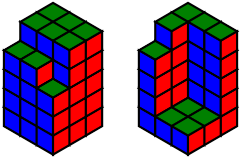

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- induction (ex.) When adding in 235, effective resistance says

$$\frac{\text{trees in new complex}}{\text{trees in original complex}} = \frac{1+2}{1} \frac{1+2+3}{1+2} \frac{1+\cdots+5}{1+\cdots+4}$$

Shifted complexes

Definition

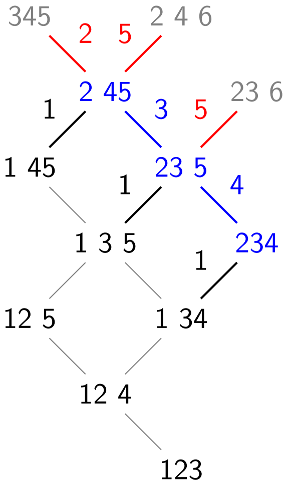
A shifted complex is a simplicial complex with:

- vertex set $1, \ldots, n$;
- if $v < w$, then you can always replace w by v.

Example ($\langle 245 \rangle$)

$123, 124, 125, 134, 135, 145, 234, 235, 245$

Enumerating spanning trees of shifted complexes



Proved by D.-Klivans-Martin '09; here are ideas of new proof (DKLM) with effective resistance

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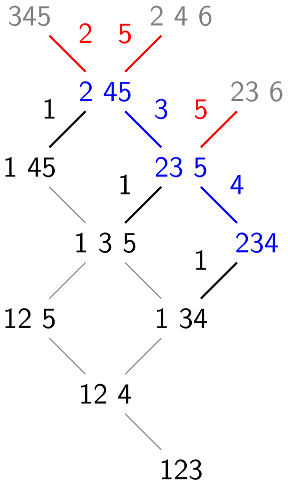

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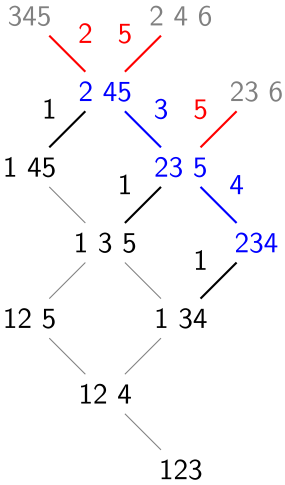

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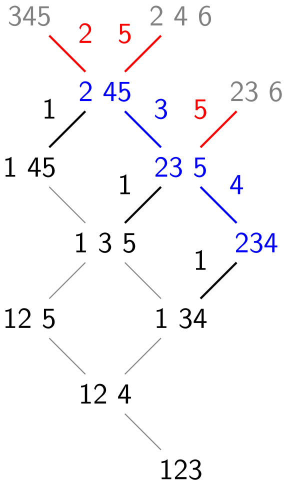
- When adding (e.g.) 23 5, effective resistance says

$$\frac{D_2 D_3 \ D_5}{D_1 D_2 \ D_4} = \frac{D_3}{D_1} \frac{D_5}{D_4}$$

where $D_j = x_1 + \cdots + x_j$.

Enumerating spanning trees of shifted complexes



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- When done, left with red edges divided by black edges with 1's.

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- Entire complex can be built one facet at a time, so that at each stage, the partially built complex is still in the family
- Effective resistance when adding each facet is a nice ratio, perhaps "covered by" divided by "covering"