Matroids and statistical dependency

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Set dependence

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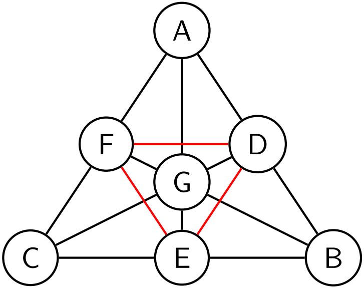
- We might expect to get any sort of simplicial complex (subsets of independent sets are independent).
- We can even get the Fano plane: A, B, C independent, $D = AB, E = BC, F = CA, G = DEF$.



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- In regression modeling, matroid structures could be used as a variable selection procedure to find the most parsimonious set of X's to predict a Y. The results of the matroid circuits would also inform which interactions ($x_1 x_2$ products) should be investigated for inclusion to the model.
- In big data settings, a matroid would identify maximally independent sets [bases] so that multiplicity can be corrected at the circuit level rather than the full data set.

How to picture data

Each variable is a vector, whose components are measurements of this variable.

- m different variables
- n different trials
- m vectors in $\mathbb{R}^n$

Example

Three variables, four trials

$$
\begin{aligned}
X &= \begin{pmatrix} 3.1 & 1 & 4 & 2 \end{pmatrix} \\
Y &= \begin{pmatrix} 2 & 1 & 6.9 & 8 \end{pmatrix} \\
Z &= \begin{pmatrix} 5 & 2.1 & 11 & 9.9 \end{pmatrix}
\end{aligned}
$$

How to measure dependence

Note in previous example:

- Knowing the value of any two of X, Y, Z tells you approximately the value of the third;
- but knowing only one variable tells you nothing about either of the others.

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How can we identify statistically independent sets in general? And capture non-linear dependence? What is "close enough"?

We will use

- Effective dependence
- Joint cumulants

These appear to be consistent measures of dependence.

Effective dependence

Effective dependence = 1 - Ψ, where

$$\Psi = \frac{|\det \Sigma|^{1/m}}{(\sum \lambda_i)/m} = \frac{\text{geometric mean}}{\text{arithmetic mean}}$$

is sphericity;

- Σ is covariance matrix (pairwise covariance of variables);
- λ_i are eigenvalues of Σ.

Joint cumulants

Definition

$$\prod_{a=1}^{b(\tau)} E(\prod_{i\in\tau_a} X_i) = \sum_{\sigma\leq\tau} \kappa_\sigma$$

By Möbius inversion, we can solve for κ's.

Example

$$E(X_1)E(X_2)E(X_3)E(X_4) = \kappa_{1|2|3|4}$$

$$E(X_1X_2)E(X_3)E(X_4) = \kappa_{1|2|3|4} + \kappa_{12|3|4}$$

So $\kappa_{12|3|4} = (E(X_1X_2) - E(X_1)E(X_2))E(X_3)E(X_4)$

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Our test of set dependence: If there is a partition of a set into two parts such that there is a cumulant dependence $\kappa_{\alpha|\beta} \neq 0$.

Matroids

Matroids make abstract ideas of independence, and model

- linear independence and dependence of sets of vectors in linear algebra;
- independent (cycle-free) sets of edges in graphs;
- etc.

When does our notion of statistical independence and dependence of sets of variables also lead to a matroid?

Closure axioms

A matroid on ground set E may be defined by closure axioms:

$$\mathrm{cl}\colon 2^E \to 2^E$$

- Closure axioms
  - $A \subseteq \mathrm{cl}(A)$
  - If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
  - $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
- Exchange axiom: If $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$, then $y \in \mathrm{cl}(A \cup x)$

For us, $x \in \mathrm{cl}(A)$ means that knowing the values of all the variables in A implies knowing something about the value of x. (Sort of: x is a function of A, with statistical noise and fuzziness.)

Invertibility

Exchange axiom: If $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$, then $y \in \mathrm{cl}(A \cup x)$

- $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$ means that in using $A \cup y$ to determine x, we must use (can't ignore) y. ("model parsimony")
- $y \in \mathrm{cl}(A \cup x)$ means we can "solve" for y in terms of x and A. (This is sort of invertibility.)

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Easiest way for a function (only way for continuous function) to be invertible is to be monotone in each variable. Fortunately, a common statistical assumption:

Definition (PRDS)

(Positive regression dependency on each one from a subset.) For any increasing set D, for for each $i \in I_0$, $P(\mathbf{X} \in D | X_i = x)$ is nondecreasing in x.

Composition

Closure axioms

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Example

When $A = x$ is a single element and $\mathrm{cl}(x) = \{x, y\}$. We need to avoid $z \in \mathrm{cl}\{x, y\}$, but $z \neq x, y$. In other words, z depends on y, and y depends on x should mean that z depends on x directly. This is a kind of transitivity.

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More generally, if Z is determined by $Y_1, \ldots, Y_p$, and each Y_i is determined by $X_1, \ldots, X_q$, then Z should be determined directly by $X_1, \ldots, X_q$. This is a kind of composition.

Remark

PRDS means the dependence will be strong enough to guarantee transitivity, and more generally composition.

Dependence axioms

How we actually show, via cumulants, that we have a matroid.
The dependent sets $\mathcal{D}$ in a matroid satisfy:

- $\emptyset \notin \mathcal{D}$
- If $D \in \mathcal{D}$ and $D' \supseteq D$, then $D' \in \mathcal{D}$
- If $I \notin \mathcal{D}$ but $I \cup x, I \cup y \in \mathcal{D}$, then $(I - z) \cup \{x, y\} \in \mathcal{D}$ for all $z \in I$.