Equivalence relations in mathematics, K-16+

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- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?
- Would we use something else in another situation, or should we always use $\frac{10}{15}$?

Equivalent fractions

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It's easy to check $\sim$ is an equivalence relation, so we can partition fractions as follows:

$\frac{a}{b}$ and $\frac{c}{d}$ are in the same part ("equivalence class") if $\frac{a}{b} \sim \frac{c}{d}$.

$\frac{1}{2}$	$\frac{17}{34}$	$\frac{2}{3}$	$\frac{10}{15}$	$\frac{1}{5}$	$\frac{3}{15}$	$\frac{4}{7}$	$\frac{20}{35}$
$\frac{4}{8}$	$\frac{6}{12}$	$\frac{4}{6}$	$\frac{14}{21}$	$\frac{10}{50}$	$\frac{8}{40}$	$\frac{40}{70}$	$\frac{16}{28}$
$\frac{10}{20}$	$\frac{7}{14}$	$\frac{20}{20}$	$\frac{8}{12}$	$\frac{2}{10}$	$\frac{7}{35}$	$\frac{8}{14}$	$\frac{36}{63}$

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- In other settings, we stick to the fraction in lowest terms, a distinguished representative.

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  - equality

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- Different situations call for different interpretations of when two shapes are “the same”.

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- At parking meters, quarters good, everything else bad.
- Everywhere, pennies bad.

Fractions, again

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When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids and no knife.

Where else do we see this?

Where else do we see this?

Glad you asked

Regrouping

To do multidigit addition and subtraction,

$$436 = 400 + 30 + 6 = 400 + 20 + 16 = 300 + 130 + 6 = \cdots$$

- Different representations are better or worse for different addition and subtraction problems.
- Using base-10 blocks, these all make different (but "equivalent") pictures.

"Unique" factorization

Completely factor 60, as

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- Distinguished representative is usually to arrange primes from smallest to largest.
- In context of factorization, 6×10 and 4×15 are different, even though usually $6 \times 10 = 4 \times 15$.

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- So it's close enough for everything we do.
- And allowing it (and all its infinite process buddies) allows us to say things like $\sqrt{2}$ and e are numbers, on the number line.

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- But it is not so obvious when expressions are equivalent.
- There are many different ideas of "distinguished representative".

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- Having the same solution set for [all relevant variables] is an equivalence relation.
- The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$,

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- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
- Usually, the distinguished representative (ordered list) to represent a combination (unordered list) is to put the items "in order"; for instance: $\{1, 10, 17, 23, 42, 54\}$.

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- This equivalence relation respects vector addition and scalar multiplication.

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- Example: Last digit arithmetic ($m = 10$).

Anti-differentiation

Solve

$$f'(x) = 3x^2$$

- "Answer" is $x^3 + C$.
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a constant.
- This equivalence relation respects addition, multiplication by a constant, which is why those are easy to deal with in anti-differentiation.

Linear Differential equations

Solve

$$y''' - 5y'' + y' - y = 3x^2$$

- Solutions of the form

  $$y = y_0 + y_p$$

  where y_0 is the general solution to the homogeneous equation, and y_p is a particular solution.
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Similarly for the matrix equation

$$Mx = b.$$

Gaussian elimination in matrices

- Consists of a series of elementary row operations that do not change the solution set.
- So at the end, we have a nicer representative of the same equivalence class (of systems with the same solution).

Cardinality

What is the cardinality of a set?

- It's not defined as a function, *per se*
- We just say when two sets have the same cardinality.
- That's an equivalence relation, not a function.
- There are some distinguished representatives: $0; 1; 2; \ldots; \mathbb{N}; \mathbb{R}$.

Why do some equivalence relations respect addition?

What we really need is to make sure that [0] acts like the additive identity:

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Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

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Similarly, the nonabelian case gives rise to normal subgroups.

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for all $[x]$.
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