Counting topologies of metric holomorphic polynomial field with simple zeros

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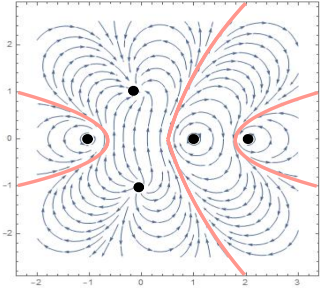
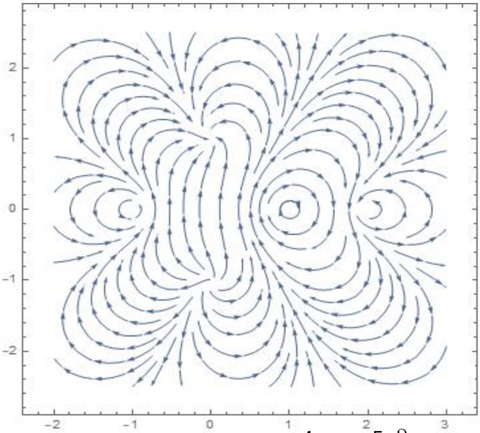
Setting the scene: Trees from flow diagrams

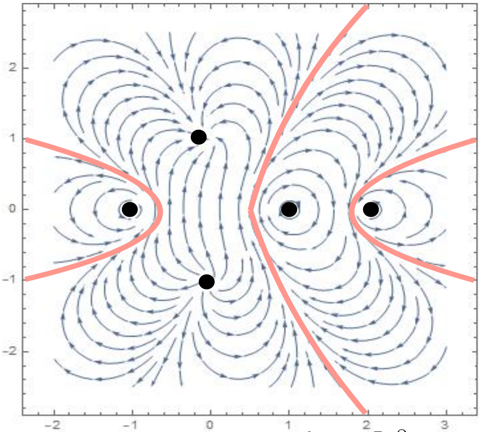


Metric holomorphic polynomial field with simple zeros



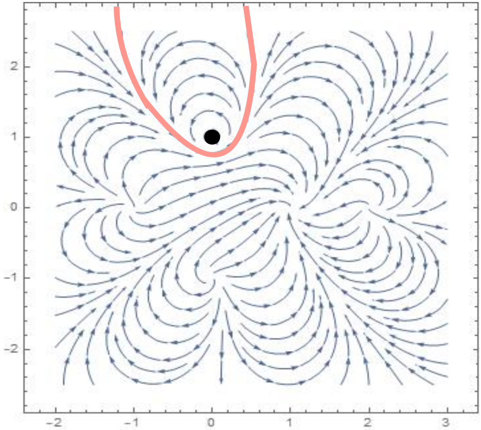
Phase portrait of $X(z) = 2i - iz - 2iz^4 + iz^5 \frac{\partial}{\partial z}$

Metric holomorphic polynomial field with simple zeros



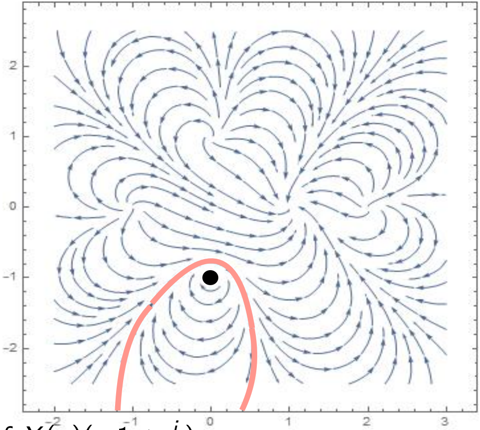
Phase portrait of $X(z) = 2i - iz - 2iz^4 + iz^5 \frac{\partial}{\partial z}$

Complex rotation



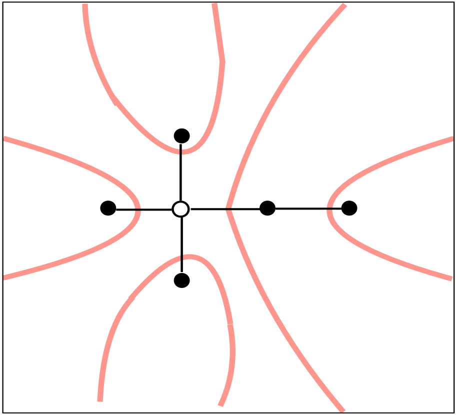
Phase portrait of $X(z)(\frac{1}{2} - i)$

Complex rotation



Phase portrait of $X(z)(-1 + \frac{i}{2})$

Put it all together, and get a graph



Trees

So we are looking at unlabeled trees with black and white vertices

- no white vertices are adjacent to each other
- each white vertex is adjacent to at least three black vertices
- no restriction on neighbors of black vertices

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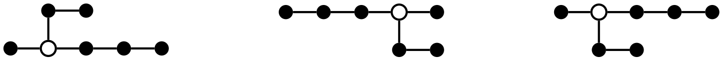
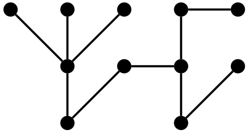
We want to count such trees up to rotation (but not reflection)

Example

The first two are the same, but the third is different.



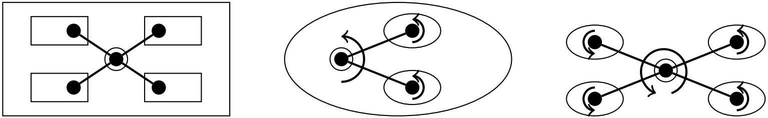
Flashback: Counting (unlabeled) trees



How to grow different kinds of rooted trees, recursively

- Rooted trees:
  - $A = X \cdot E(A)$,
  - E stands for "set of"
- Ordered rooted tree:
  - $A_L = X \cdot L(A_L)$
  - L stands for "linear order"
- Planar rooted trees:
  - $P = X + X \cdot C(A_L)$
  - C stands for "cyclic order"

Example



How to count different kinds of rooted trees (species)

Species and exponential generating functions

- Rooted trees:
  - $\mathcal{A} = X \cdot E(\mathcal{A})$,
  - E stands for "set of"
  - $E(x) = e^x$
- Ordered rooted tree:
  - $\mathcal{A}_L = X \cdot L(\mathcal{A}_L)$
  - L stands for "linear order"
  - $L(x) = \frac{1}{1-x}$
- Planar rooted trees:
  - $\mathcal{A}_P = X + X \cdot C(\mathcal{A}_L)$
  - C stands for "cyclic order"
  - $C(x) = -\log(1-x)$

These give recursive equations we can solve.

Unrooting I: Center of tree

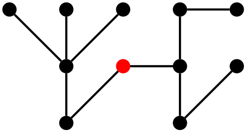
Definition

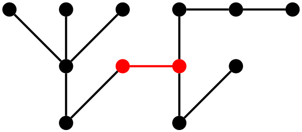
Center of a tree is the set of vertices v that minimize

$$\max_u \mathrm{d}(u, v)$$

It is always either a single vertex, or an edge.

Example





Unrooting I: Center of tree

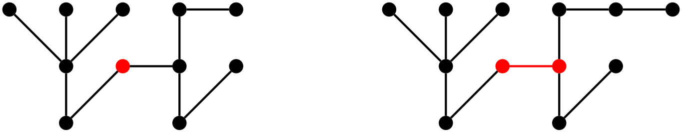
Definition

Center of a tree is the set of vertices v that minimize

$$\max_u \mathrm{d}(u, v)$$

It is always either a single vertex, or an edge. So this naturally roots a tree at either a vertex or an edge.

Example



Unrooting II: Dissymmetry theorem

Theorem (Dissymmetry)

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

where a denotes unrooted trees and E_2 is the species of sets with exactly two elements.

Unrooting II: Dissymmetry theorem

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$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

where $\mathfrak{a}$ *denotes unrooted trees and* E_2 *is the species of sets with exactly two elements.*

Proof.

(Sketch) LHS is trees rooted at a vertex or an edge. RHS is trees (unrooted) or ordered pair of trees.

Unrooting II: Dissymmetry theorem

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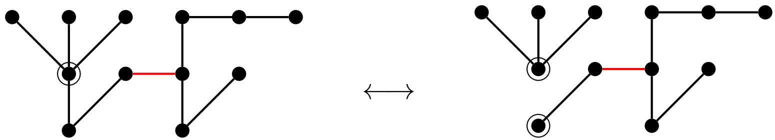
$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

where $\mathfrak{a}$ denotes unrooted trees and E_2 is the species of sets with exactly two elements.

Proof.

(Sketch) LHS is trees rooted at a vertex or an edge. RHS is trees (unrooted) or ordered pair of trees.So we need isomorphism between trees rooted at vertex or edge other than the center, with ordered pairs of rooted trees.



Isomorphism type (removing the labels)

Definition

Type generating series of species F is ordinary (non-exponential) generating function of isomorphism types of F. But to actually compute, we need:

$$\tilde{F}(x) = Z_F(x, x^2, x^3, \ldots)$$

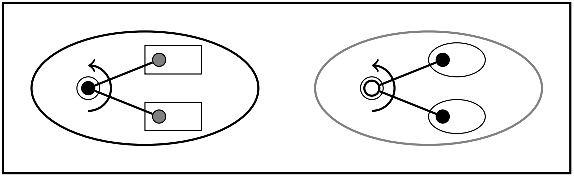
where Z_F is the cycle index series (definition suppressed)

Example

$$Z_C(x_1, x_2, x_3, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - x_k}$$

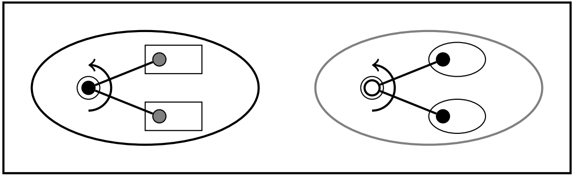
(others not so bad)

Return to the present day: Counting our trees



Black and white vertices, not at the root

Similar to ordered rooted trees, but now color-aware



$$Y_1 = X_1 \cdot L(Y_1 + Y_2) \qquad Y_2 = X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_1 + Y_2))$$

$$Y_3 = Y_1 + Y_2 = X_1 \cdot L(Y_3) + X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_3))$$

Recursive equation

$$Y_3 = Y_1 + Y_2 = X_1 \cdot L(Y_3) + X_2 \cdot L_{\geq 2}(X_1 \cdot L(Y_3))$$

$$y_3 = x_1\ell + x_2\frac{(x_1\ell)^2}{1-(x_1\ell)}$$

where $\ell = \frac{1}{1-y_3}$.

Recursive equation

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$$y_3 = x_1\ell + x_2 \frac{(x_1\ell)^2}{1 - (x_1\ell)}$$

where $\ell = \frac{1}{1-y_3}$. Simplifying,

$$x_1 + x_1^2(x_2 - 1) - (y_3 - 1)^2 y_3 - x_1 y_3^2 = 0.$$

Unique real root $y_3(x_1, x_2) =$

Recursive equation

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Unique real root $y_3(x_1, x_2) =$

$$\frac{\frac{2-x1}{3} + \left(2^{1/3}\left(-1+4x1-x1^2\right)\right)}{\left(3\left(2-12x1+15x1^2+2x1^3-27x1^2x2+\sqrt{\left(4(-1+4x1-x1^2)^3+(2-12x1+15x1^2+2x1^3-27x1^2x2)^2\right)}\right)^{1/3}\right)}$$

$$-\frac{1}{3\ 2^{1/3}}\left(2 - 12x1 + 15x1^2 + 2x1^3 - 27x1^2x2 + \sqrt{\left(4\left(-1+4x1-x1^2\right)^3 + \left(2 - 12x1 + 15x1^2 + 2x1^3 - 27x1^2x2\right)^2\right)}\right.$$

Dissymmetry again

Recall

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

The same arguments apply. But now, paying attention to color,

$$A \approx (X_1 \cdot (1 + C(Y_3))) + (X_2 \cdot C_{\geq 3}(X_1 \cdot L(Y_3)))$$

Dissymmetry again

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$$E_2(A) \approx E_2(Y_1) + Y_2 \cdot Y_1 = E_2(Y_3) - E_2(Y_2)$$

Dissymmetry again

Recall

$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

The same arguments apply. But now, paying attention to color,

$$\begin{aligned} A &\approx (X_1 \cdot (1 + C(Y_3))) + (X_2 \cdot C_{\geq 3}(X_1 \cdot L(Y_3))) \\ E_2(A) &\approx E_2(Y_1) + Y_2 \cdot Y_1 = E_2(Y_3) - E_2(Y_2) \\ A^2 &\approx Y_1^2 + 2Y_1Y_2 = Y_3^2 - Y_2^2 \end{aligned}$$

Dissymmetry again

Recall

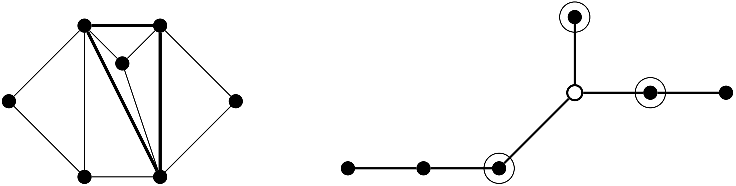
$$\mathcal{A} + E_2(\mathcal{A}) = \mathfrak{a} + \mathcal{A}^2,$$

The same arguments apply. But now, paying attention to color,

$$\begin{aligned} A &\approx (X_1 \cdot (1 + C(Y_3))) + (X_2 \cdot C_{\geq 3}(X_1 \cdot L(Y_3))) \\ E_2(A) &\approx E_2(Y_1) + Y_2 \cdot Y_1 = E_2(Y_3) - E_2(Y_2) \\ A^2 &\approx Y_1^2 + 2Y_1 Y_2 = Y_3^2 - Y_2^2 \end{aligned}$$

This can be stated more generally for "multi-sort" species.
(And then, to remove labels, again bring in cycle index series.)

Aftermath: Data and Specializations



Data

$$x_1 + x_1^2 + x_1^3 + x_2x_1^3 + 2x_1^4 + 2x_2x_1^4 + 3x_1^5 + 5x_2x_1^5 + x_2^2x_1^5 + 6x_1^6 + 16x_2x_1^6 + 5x_2^2x_1^6 + 14x_1^7 + 48x_2x_1^7 + 30x_2x_1^7 + 2x_2^3x_1^7 + 34x_1^8 + 164x_2x_1^8 + 146x_2^2x_1^8 + 20x_2^3x_1^8 + 95x_1^9 + 559x_2x_1^9 + 693x_2^2x_1^9 + 175x_2^3x_1^9 + 7x_2^4x_1^9 + 280x_1^{10} + 1952x_2x_1^{10} + 3108x_2^2x_1^{10} + 1254x_2^3x_1^{10} + 95x_2^4x_1^{10} + 854x_1^{11} + 6872x_2x_1^{11} + 13608x_2^2x_1^{11} + 7752x_2^3x_1^{11} + 1125x_2^4x_1^{11} + 19x_2^5x_1^{11} + 2694x_1^{12} + 24520x_2x_1^{12} + 58200x_2^2x_1^{12} + 44112x_2^3x_1^{12} + 10108x_2^4x_1^{12} + 480x_2^5x_1^{12} 8714x_1^{13} + 88006x_2x_1^{13} + 245322x_2^2x_1^{13} + 235557x_2^3x_1^{13} + 77580x_2^4x_1^{13} + 7084x_2^5x_1^{13} + 86x_2^6x_1^{13} + \cdots$$

	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	2	3	6	14	34	95	280	854	2694
1	0	0	1	2	5	16	48	164	559	1952	6872	24520
2	0	0	0	0	1	5	30	146	693	3108	13608	58200
3	0	0	0	0	0	0	2	20	175	1254	7752	44112
4	0	0	0	0	0	0	0	0	7	95	1125	10108
5	0	0	0	0	0	0	0	0	0	0	19	480
6	0	0	0	0	0	0	0	0	0	0	0	0

No white vertices

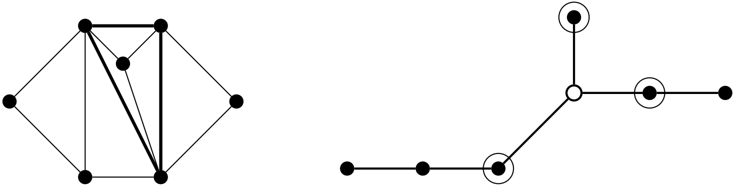
	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	2	3	6	14	34	95	280	854	2694	8714

Unlabeled plane trees

One white vertex

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	1	2	5	16	48	164	559	1952	6872	24520

Triangulations of an *n*-gon with exactly one internal vertex.
(Brown, 1964)



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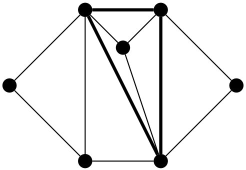

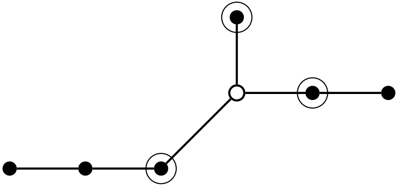

Both are circular orders of (at least three) Catalan-things (ordered rooted trees or rooted triangulations).

Minimal black vertices

	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1												
1	0	0	1										
2	0	0	0	0	1								
3	0	0	0	0	0	0	2						
4	0	0	0	0	0	0	0	0	7				
5	0	0	0	0	0	0	0	0	0	0	19		
6	0	0	0	0	0	0	0	0	0	0	0	0	86

Unlabeled 3-gonal cacti with *n* triangles. (Bóna, Bousquet, Labelle, Leroux, 2000)

To get their graphs from ours in this case, connect all black vertices adjacent to the same white vertex, remove all white vertices.