Cuts and flows in cell complexes

Art Duval[1] Caroline Klivans[2] Jeremy Martin[3]

[1]University of Texas at El Paso

[2]Brown University

[3]University of Kansas

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Critical groups, cuts, and flows

Theorem (Bacher, de la Harpe, Nagnibeda)

$$K(G) \cong \mathcal{C}^\sharp/\mathcal{C} \cong \mathcal{F}^\sharp/\mathcal{F} \cong \mathbb{Z}^{|E|}/(\mathcal{C} \oplus \mathcal{F})$$

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Theorem (DKM)

$$0 \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C} \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma, \mathbb{Z})) \to 0$$
$$0 \to \mathbf{T}(\tilde{H}^d(\Sigma, \mathbb{Z})) \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F} \to 0$$

where Σ is a d-dimensional cell complex, $K(\Sigma)$ is its critical group, $K^*(\Sigma)$ is its cocritical group, $\mathcal{C}$ is the cut lattice, $\mathcal{F}$ is the flow lattice, and **T** *denotes torsion (finite) part of an abelian group.*

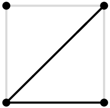
Cuts and bonds

Let G be a connected graph

Definition

A cut is a collection of edges in G whose removal disconnects the graph;

Example



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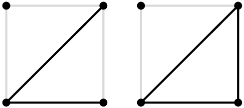

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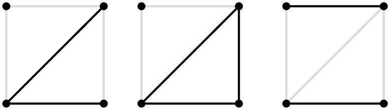

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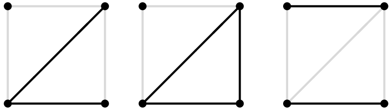

Remark

Using matroid language, bonds are cocircuits.

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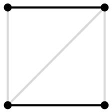

1	0	0	1	0
−1	1	0	0	1
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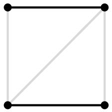

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Question

What is a basis?

Fundamental bond

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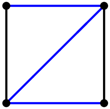

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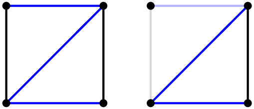

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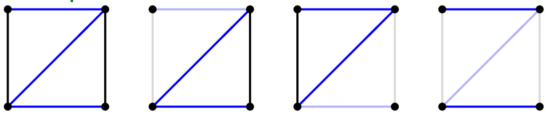

Theorem

For a fixed spanning tree, the collection of fundamental bonds forms a basis of cut space

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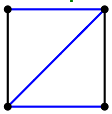

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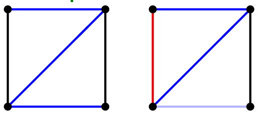

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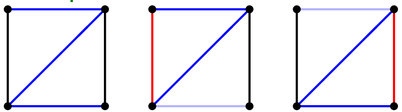

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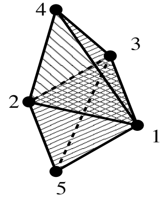
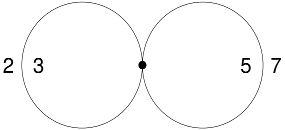
Think the boundary of each facet being a $\mathbb{Z}$-linear combination of ridges.

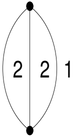
Remark

Any $\mathbb{Z}$ matrix can be the boundary matrix of a cell complex

Examples







$$\begin{matrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 5 & 7 \end{matrix}$$

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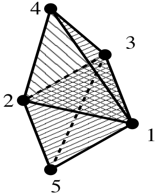
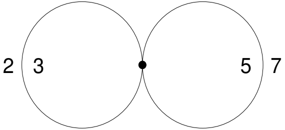
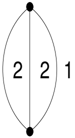
Cellular matroids

- Matroid whose elements are columns of boundary matrix







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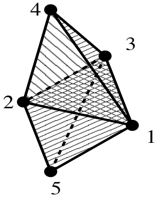

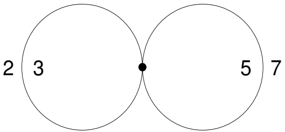

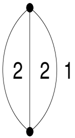

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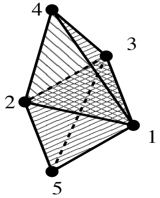

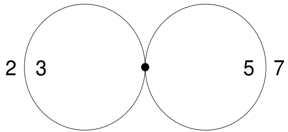

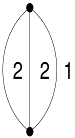

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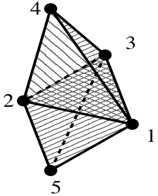

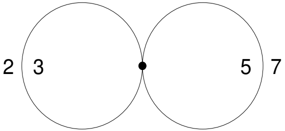

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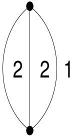

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i-dimensional cut space of cell complex X is

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Bonds are the cocircuits of cellular matroid

Topological interpretation of bonds

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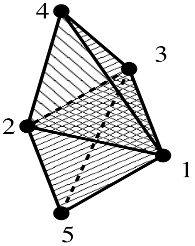
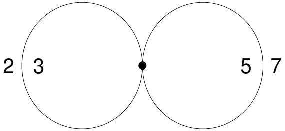
Bonds are minimal for increasing $(i-1)$-dimensional homology instead of decreasing i-dimensional homology

Examples





Characteristic vectors of bonds

Fix bond B

Proposition

$\mathrm{Cut}_B(X) := (\{0\} \cup (\mathrm{Cut}_i(X) \cap \{v : \mathrm{supp}(v) = B\}))$ *is 1-dimensional*

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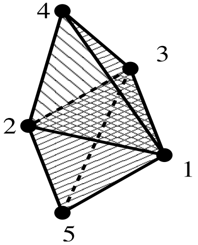

Topological interpretation of characteristic vector

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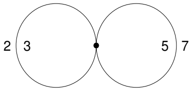

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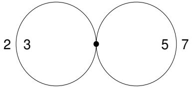

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Let A be a cellular spanning forest of X/B. Then $\mathrm{Cut}_B(X)$ is spanned by

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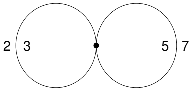

If $B = \{F_5, F_7\}$, then $\chi(B, F_2) = 2(5F_5 + 7F_7)$, but $\chi(B, F_3) = 3(5F_5 + 7F_7)$.

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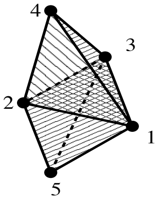

$\Upsilon = \{124, 134, 123, 135, 235\}$

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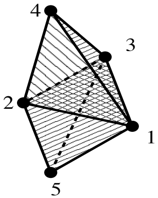

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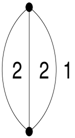

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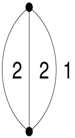

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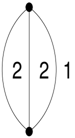

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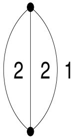
Topological interpretation of characteristic vector

Example



$$\begin{matrix} 0 & -2 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{matrix} ;\ \tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2);\ \chi(C) = (4, 2, 2)$$

Theorem (DKM)

$$\chi(C) = \sum_{F \in C} \pm |\mathbf{T}\tilde{H}(C \setminus F, \mathbb{Z})| F$$

spans $\mathrm{Cut}_C(X)$, *where* $\mathbf{T}$ *stands for torsion part.*

Definition

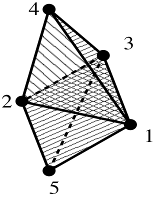
The characteristic vector of C is $\chi(C)$

Fundamental circuit

Definition

Given a spanning forest Υ and an face $F \notin \Upsilon$, the fundamental circuit is the unique circuit in $\Upsilon \cup \{F\}$.

Example



$\Upsilon = \{124, 134, 123, 135, 235\}$

F	C
234	$\{123, 124, 134, 234\}$
235	$\{123, 125, 135, 235\}$

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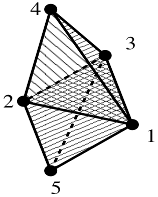

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Theorem (DKM)

For a fixed spanning forest, the collection of characteristic vectors of fundamental circuits forms a basis of flow space