Weighted spanning tree enumerators of color-shifted complexes

Ghodratollah Aalipour[1,2] Art Duval[1]

[1]University of Texas at El Paso

[2]Sharif University of Technology

CombinaTexas
Texas A&M University
April 19, 2014

Spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

$T \subseteq E(G)$ is a **spanning tree** of G when:

0. spanning: T contains all vertices;
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T}(\prod_{v \in e} x_v)$.

Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \operatorname{wt} T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\operatorname{wt} T = \prod_{e \in T} \operatorname{wt} e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

Example (K_4)

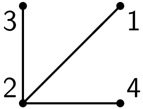
Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where wt $T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

Example (K_4)

- 4 trees like: $T =$



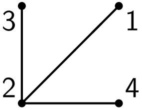
wt $T = (x_1 x_2 x_3 x_4) x_2^2$

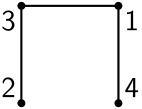
Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where wt $T = \prod_{e \in T} \text{wt } e = \prod_{e \in T}(\prod_{v \in e} x_v)$.

Example (K_4)

- 4 trees like: $T =$  wt $T = (x_1x_2x_3x_4)x_2^2$

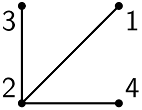
- 12 trees like: $T =$  wt $T = (x_1x_2x_3x_4)x_1x_3$

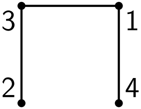
Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

Example (K_4)

- 4 trees like: $T =$ 

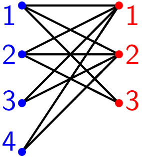
$\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2$
- 12 trees like: $T =$ 

$\text{wt } T = (x_1 x_2 x_3 x_4) x_1 x_3$
- Total is $(x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2$.

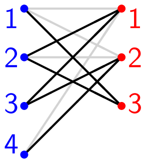
Ferrers graphs (Ehrenborg-van Willigenburg '04)

Example ($\langle 42, 23 \rangle$)

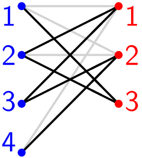
	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		



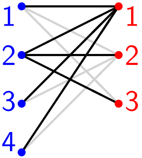
Spanning trees of Ferrers graphs

- $T =$  $\text{wt}\, T = (1234)(123)23123$

Spanning trees of Ferrers graphs

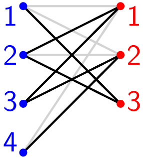
- $T =$ 

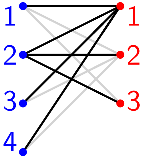
$\text{wt } T = (1234)(123)23123$

- $T =$ 

$\text{wt } T = (1234)(123)2^2 1^3$

Spanning trees of Ferrers graphs

- $T =$ 

wt $T = (1234)(123)23123$
- $T =$ 

wt $T = (1234)(123)2^2 1^3$
- Total is $(1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$

Theorem

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		

Total is (1234)(123)

Theorem

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		

Total is $(1234)(123)(1+2+3+4)(1+2)$

Theorem

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		

Total is $(1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$

Theorem

	1	2	3	4
1	11	21	31	41
2	12	22	32	42
3	13	23		

Total is $(1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$

Theorem (Ehrenborg-van Willigenburg)

This works in general

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$$D(G) = \mathrm{diag}(\deg v_1, \ldots, \deg v_n)$$

$$A(G) = \text{adjacency matrix}$$

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$$D(G) = \mathrm{diag}(\deg v_1, \ldots, \deg v_n)$$

$$A(G) = \text{adjacency matrix}$$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ *spanning trees*

Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$$D(G) = \operatorname{diag}(\deg v_1, \ldots, \deg v_n)$$

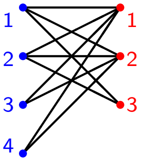
$$A(G) = \text{adjacency matrix}$$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

"Reduced": remove rows/columns corresponding to any one vertex

Example $\langle 42, 23 \rangle$

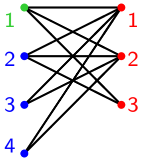


$\partial =$

	11	12	13	21	22	23	31	32	41	42
1	-1	-1	-1	0	0	0	0	0	0	0
2	0	0	0	-1	-1	-1	0	0	0	0
3	0	0	0	0	0	0	-1	-1	0	0
4	0	0	0	0	0	0	0	0	-1	-1
1	1	0	0	1	0	0	1	0	1	0
2	0	1	0	0	1	0	0	1	0	1
3	0	0	1	0	0	1	0	0	0	0

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Example $\langle 42, 23 \rangle$



$\partial =$

	11	12	13	21	22	23	31	32	41	42
1	-1	-1	-1	0	0	0	0	0	0	0
2	0	0	0	-1	-1	-1	0	0	0	0
3	0	0	0	0	0	0	-1	-1	0	0
4	0	0	0	0	0	0	0	0	-1	-1
1	1	0	0	1	0	0	1	0	1	0
2	0	1	0	0	1	0	0	1	0	1
3	0	0	1	0	0	1	0	0	0	0

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L_r = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Example $\langle 42, 23 \rangle$

$\partial =$

	11	12	13	21	22	23	31	32	41	42
1	-1	-1	-1	0	0	0	0	0	0	0
2	0	0	0	-1	-1	-1	0	0	0	0
3	0	0	0	0	0	0	-1	-1	0	0
4	0	0	0	0	0	0	0	0	-1	-1
1	1	0	0	1	0	0	1	0	1	0
2	0	1	0	0	1	0	0	1	0	1
3	0	0	1	0	0	1	0	0	0	0

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L_r = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$\det(L_r) = 96$, the number of spanning trees of $\langle 42, 23 \rangle$.

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \mathrm{wt}\, T = |\det \hat{L}_r(G)|,$$

where $\hat{L}_r(G)$ is reduced weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$\hat{D}(G) = \mathrm{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$\hat{A}(G) =$ adjacency matrix

(entry $x_i x_j$ for edge $v_i v_j$)

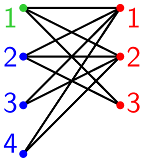
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G) =$ incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example ($\langle 42, 23 \rangle$)



$$\hat{L}_r = \begin{pmatrix} 2(1+2+3) & 0 & 0 & -21 & -22 & -23 \\ 0 & 3(1+2) & 0 & -31 & -32 & 0 \\ 0 & 0 & 4(1+2) & -41 & -42 & 0 \\ -21 & -31 & -41 & 1(1+2+3+4) & 0 & 0 \\ -22 & -32 & -42 & 0 & 2(1+2+3+4) & 0 \\ -23 & 0 & 0 & 0 & 0 & 3(1+2) \end{pmatrix}$$

$$\det \hat{L}_r = (1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$$

Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete d-dimensional simplicial complex on n vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third.
- When $d = 1$, coincides with usual definition.

Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete d-dimensional simplicial complex on n vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third.
- When $d = 1$, coincides with usual definition.

Example

$n = 5, d = 2 : \Upsilon = \{123, 124, 125, 134, 135, 245\}$

Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^d)} \qquad = n^{\binom{n-2}{d}}$$

Counting simplicial spanning trees of K_n^d

Theorem [Kalai '83]

$$\tau(K_n^d) = \sum_{\Upsilon \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}$$

Weighted simplicial spanning trees of K_n^d

As before,

$$\operatorname{wt}\Upsilon = \prod_{F\in\Upsilon} \operatorname{wt} F = \prod_{F\in\Upsilon} \Big(\prod_{v\in F} x_v \Big)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\operatorname{wt}\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Weighted simplicial spanning trees of K_n^d

As before,

$$\operatorname{wt}\Upsilon = \prod_{F\in\Upsilon} \operatorname{wt} F = \prod_{F\in\Upsilon}\Big(\prod_{v\in F} x_v\Big)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$
$$\operatorname{wt}\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Theorem (Kalai, '83)

$$\hat{\tau}(K_n^d) := \sum_{T\in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 (\operatorname{wt}\Upsilon)$$
$$= (x_1\cdots x_n)^{\binom{n-2}{d-1}}(x_1+\cdots+x_n)^{\binom{n-2}{d}}$$

Simplicial spanning trees of arbitrary simplicial complexes

Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third.
- When $d = 1$, coincides with usual definition.

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_{\Gamma},$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = *restriction of* ∂_d *to faces not in* Γ

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = *restriction of* ∂_d *to faces not in* Γ
- *reduced Laplacian* $L_\Gamma = \partial_\Gamma \partial^T{}_\Gamma$

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = *restriction of* ∂_d *to faces not in* Γ
- *reduced Laplacian* $L_\Gamma = \partial_\Gamma \partial^T{}_\Gamma$
- *Weighted version: Multiply column* F *of* ∂ *by* x_F

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ *= restriction of* ∂_d *to faces not in* Γ
- *reduced Laplacian* $L_\Gamma = \partial_\Gamma \partial^T{}_\Gamma$
- *Weighted version: Multiply column* F *of* ∂ *by* x_F

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = *restriction of* ∂_d *to faces not in* Γ
- *reduced Laplacian* $L_\Gamma = \partial_\Gamma \partial^T{}_\Gamma$
- *Weighted version: Multiply column* F *of* ∂ *by* x_F

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Example: Octahedron

- Vertices 1, 2, 1, 2, 1, 2.
- Facets 111, 112, 121, 122, 211, 212, 221, 222,
- $\Gamma = 11, 12, 11, 12, 22$ spanning tree of 1-skeleton, so remove (rows and columns corresponding to) those edges from weighted Laplacian.
- $\det \hat{L}_\Gamma = (121212)^3(1+2)(1+2)(1+2)$.

Color-shifted complexes

Definition (Babson-Novik, '96)

A color-shifted complex is a simplicial complex with:

- vertex set $V_1 \dot\cup \ldots \dot\cup V_r$ (V_i is set of vertices of color i);
- $|V_i| = n_i$;
- every facet contains one vertex of each color; and
- if $v < w$ are vertices of the same color, then you can always replace w by v.

Note: $r = 2$ is Ferrers graphs

Example

Octahedron is $\langle 222 \rangle$

Example $\langle 235, 324, 333 \rangle$

facets

111	112	113	114	115
121	122	123	124	125
131	132	133	134	135
211	212	213	214	215
221	222	223	224	225
231	232	233	234	235
311	312	313	314	
321	322	323	324	
331	332	333		

Example ⟨235, 324, 333⟩

facets

111	112	113	114	115
121	122	123	124	125
131	132	133	134	135
211	212	213	214	215
221	222	223	224	225
231	232	233	234	235
311	312	313	314	
321	322	323	324	
331	332	333		

ridges

11	12	13		
21	22	23		
31	32	33		
11	12	13	14	15
21	22	23	24	25
31	32	33	34	
11	12	13	14	15
21	22	23	24	25
31	32	33	34	35

Example $\langle 235, 324, 333 \rangle$

facets

111	112	113	114	115
121	122	123	124	125
131	132	133	134	135
211	212	213	214	215
221	222	223	224	225
231	232	233	234	235
311	312	313	314	
321	322	323	324	
331	332	333		

reduced ridges

11	12	13		
21	22	23		
31	32	33		
11	12	13	14	15
21	22	23	24	25
31	32	33	34	
11	12	13	14	15
21	22	23	24	25
31	32	33	34	35

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\begin{aligned}
&(1^7 2^7 3^6)(1+2+3)^5(1+2)^3 \\
&\quad \times (1^7 2^6 3^6)(1+2+3)^8(1+2) \\
&\quad \times (1^5 2^5 3^5 4^5 5^4)(1+\cdots+5)^2(1+\cdots 4)(1+\cdots 3)
\end{aligned}$$

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\times\,(1^5 2^5 3^5 4^5 5^4)(1 + \cdots + 5)^2(1 + \cdots 4)(1 + \cdots 3)$$

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$\times\ (1^5 2^5 3^5 4^5 5^4)$

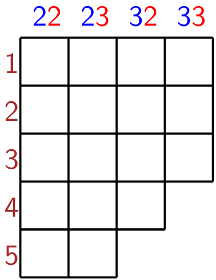
11	12	13		
21	22	23		
31	32	33		

11	12	13	14	15
21	22	23	24	25
31	32	33	34	

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35

Enumeration: $\hat{\tau}(\langle 235, 324, 333\rangle)$

$$\times \qquad (1 + \cdots + 5)^2(1 + \cdots 4)(1 + \cdots 3)$$



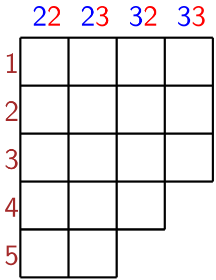
Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\times (1^5 2^5 3^5 4^5 5^4)(1 + \cdots + 5)^2 (1 + \cdots 4)(1 + \cdots 3)$$

11	12	13
21	22	23
31	32	33

11	12	13	14	15
21	22	23	24	25
31	32	33	34	

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35



Enumeration

Theorem (Aalipour-D.)

When $r = 3$, this always works.

Enumeration

Theorem (Aalipour-D.)

When $r = 3$, this always works.

Conjecture

When $r > 3$, this always works.

Enumeration

Theorem (Aalipour-D.)

When $r = 3$, this always works.

Conjecture

When $r > 3$, this always works.

Remark

The codimension-1 spanning tree will be a different tree for each color. For each color's factors, treat that color as "last".
Example: $r = 4$ (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.

Proof (via example $\langle 235, 324, 333 \rangle$)

$$\det \begin{pmatrix} 22(1+\cdot+5) & 0 & 0 & 0 & \cdots \\ 0 & 23(1+\cdot+5) & 0 & 0 & \cdots \\ 0 & 0 & 22(1+\cdot+4) & 0 & \cdots \\ 0 & 0 & 0 & 33(1+\cdot+3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= (2^2 3^2 2^2 3^2 \cdots) \det \begin{pmatrix} 1+\cdot+5 & 0 & 0 & 0 & \cdots \\ 0 & 1+\cdot+5 & 0 & 0 & \cdots \\ 0 & 0 & 1+\cdot+4 & 0 & \cdots \\ 0 & 0 & 0 & 1+\cdot+3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

By "identification of factors" (Martin-Reiner, '03), to show $(1+\cdot+5)^2$ is a factor of the det, just show nullspace of this matrix ≥ 2, when $1+\cdot+5=0$.