Weighted spanning tree enumerators of color-shifted complexes

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CombinaTexas Texas A&M University April 19, 2014

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Spanning trees of K_n

Theorem (Cayley) K_n has n^{n-2} spanning trees. $T \subseteq E(C)$ is a spanning tree of .

- $T \subseteq E(G)$ is a **spanning tree** of G when:
 - 0. spanning: T contains all vertices;
 - 1. connected $(\tilde{H}_0(T) = 0)$
 - 2. no cycles $(\tilde{H}_1(T) = 0)$
 - 3. correct count: |T| = n 1

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

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Theorem (Cayley-Prüfer)

$$\sum_{T\in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where wt $T = \prod_{e \in T}$ wt $e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

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Example (K_4)

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Example (K_4)

• 4 trees like:
$$T = 2$$
 • 4 wt $T = (x_1 x_2 x_3 x_4) x_2^2$

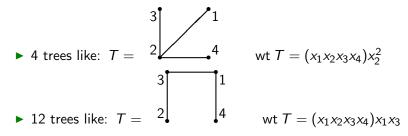
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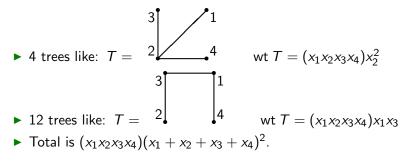
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Example (K_4)



Counting trees Matrix-tree theorem

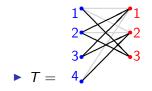
Ferrers graphs (Ehrenborg-van Willigenburg '04)

Example $(\langle 42, 23 \rangle)$ 1 2 3 4 1 11 21 31 41 2 12 22 32 42 3 13 23 4

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Counting trees Matrix-tree theorem

Spanning trees of Ferrers graphs

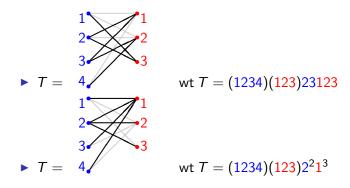


wt
$$T = (1234)(123)23123$$

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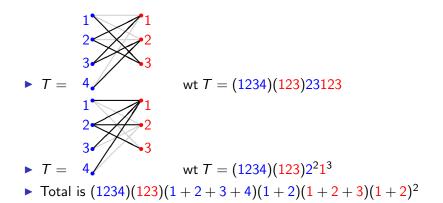
Counting trees Matrix-tree theorem

Spanning trees of Ferrers graphs



Counting trees Matrix-tree theorem

Spanning trees of Ferrers graphs



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Counting trees Aatrix-tree theorem

Theorem

	1	2	3	4
1	11	21	31	4 1
2	12	22	<mark>3</mark> 2	4 2
3	13	23		

Total is (1234)(123)

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Counting trees Matrix-tree theorem

Theorem

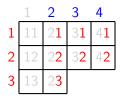
	1	2	3	4
1	11	21	31	41
2	1 2	2 2	3 2	4 2
3	1 3	2 3		

Total is (1234)(123)(1+2+3+4)(1+2)

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Counting trees Matrix-tree theorem

Theorem



Total is $(1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$

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Counting trees Matrix-tree theorem

Theorem

	1	2	3	4
1	11	21	31	4 1
2	12	22	<mark>3</mark> 2	4 2
3	13	23		

Total is $(1234)(123)(1 + 2 + 3 + 4)(1 + 2)(1 + 2 + 3)(1 + 2)^2$ Theorem (Ehrenborg-van Willigenburg) This works in general

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Theorem (Kirchoff's Matrix-Tree) G has $|\det L_r(G)|$ spanning trees **Definition** The Laplacian matrix of graph G, denoted by L (G).

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Theorem (Kirchoff's Matrix-Tree) *G* has $|\det L_r(G)|$ spanning trees **Definition** The Laplacian matrix of graph *G*, denoted by *L* (*G*). Defn 1: L(G) = D(G) - A(G) $D(G) = \text{diag}(\deg v_1, \dots, \deg v_n)$ A(G) = adjacency matrix

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Theorem (Kirchoff's Matrix-Tree) G has $|\det L_r(G)|$ spanning trees **Definition** The reduced Laplacian matrix of graph G, denoted by $L_r(G)$. Defn 1: L(G) = D(G) - A(G) $D(G) = \text{diag}(\text{deg } v_1, \ldots, \text{deg } v_n)$ A(G) = adjacency matrixDefn 2: $L(G) = \partial(G)\partial(G)^T$ $\partial(G) =$ incidence matrix (boundary matrix) "Reduced": remove rows/columns corresponding to any one vertex

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Counting trees Matrix-tree theorem

Example $\langle \mathbf{42},\mathbf{23}\rangle$

					11	12	13	21	22	<mark>23</mark>	31	<mark>3</mark> 2	41	4 2
	$\prec 1$	1		1	-1	-1	-1	0	0	0	0	0	0	0
2	\rightarrow	2		2	0	0	0	-1	-1	-1	0	0	0	0
- 2	XX	_	$\partial =$	3	0	0	0	0	0	0	-1	-1	0	0
3	//~	3	0 -	4	0	0	0	0	0	0	0	0	-1	-1
. //				1	1	0	0	1	0	0	1	0	1	0
4 🦨				2	0	1	0	0	1	0	0	1	0	1
				3	0	0	1	0	0	1	0	0	0	0
	/ 3	0	0	0	- :	1 –	- 1	$-1\rangle$						
	0	3	0 2	0	-1	. –	-1	-1						
	0	0	2	0	$^{-1}$. –	-1	0						
L =	0	0	0	2	$^{-1}$. –	-1	0						
	- 1	-1	-1	-1	4		0	0						
	- 1	-1	-1	-1	0		4	0						
	$\setminus -1$	-1	0	0	0		0	2 /						

Counting trees Matrix-tree theorem

Example $\langle \mathbf{42},\mathbf{23}\rangle$

			11	12	13	21	22	<mark>23</mark>	31	<mark>3</mark> 2	41	42	
		1	-1	-1	-1	0	0	0	0	0	0	0	
2		2	0	0	0	-1	-1	-1	0	0	0	0	
	$\partial =$	3	0	0	0	0	0	0	-1	-1	0	0	
3	o =	4	0	0	0	0	0	0	0	0	-1	-1	
		1	1	0	0	1	0	0	1	0	1	0	
4 🧨		2 3	0	1	0	0	1	0	0	1	0	1	
		3	0	0	1	0	0	1	0	0	0	0	
$L = \begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}$	3 0 2 2	0 0 2 -1 -1 0	 1 1 1 4 0 0	 	- 1 -1 -1 0 4 0	$\begin{pmatrix} -1\\ -1\\ 0\\ 0\\ 0\\ 0\\ 2 \end{pmatrix}$	$L_r =$	$\begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$	$0 \\ 2 \\ 0 \\ -1 \\ -1 \\ 0$	0 0 2 -1 -1 0	$-1 \\ -1 \\ -1 \\ 4 \\ 0 \\ 0 \\ 0$	$-1 \\ -1 \\ -1 \\ 0 \\ 4 \\ 0$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

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Counting trees Matrix-tree theorem

Example $\langle 42, 23 \rangle$

	11	12	1 3	21	22	23	31	<mark>3</mark> 2	41	42		
1	-1	-1	-1	0	0	0	0	0	0	0		
2	0	0	0	-1	-1	-1	0	0	0	0		
$\partial = \frac{3}{2}$	0	0	0	0	0	0	-1	-1	0	0		
$\mathcal{O} = 4$	0	0	0	0	0	0	0	0	-1	-1		
1	1	0	0	1	0	0	1	0	1	0		
2	0		0	0	1	0	0	1		1		
3	0	0	1	0	0	1	0	0	0	0		
(300) 0	_	1	- 1	-1		12	0	0	1	1	1
0 3 () 0	-1	L	-1	-1			2	0	-1	-1	-11
0 0 2	2 0	-1	L	-1	0		0	2	0	-1	-1	0
$L = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$) 2	-1	L	-1	0	$L_r =$	1	1	2	-1	-1	0
-1 -1 -	1 -1	1 4		0	0		1	-1	-1	4	4	0
$L = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 &$	1 -1	1 0		4	0			-1	-1	$-1 \\ -1 \\ -1 \\ 4 \\ 0 \\ 0$	4	2
$\begin{pmatrix} -1 & -1 \end{pmatrix}$ (0 0	0		0	2 /		/-1	0	0	0	0	2 /

det(L_r) = 96, the number of spanning trees of $\langle 42, 23 \rangle$.

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Counting trees Matrix-tree theorem

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \operatorname{wt} T = |\det \hat{L}_r(G)|,$$

where $\hat{L}_r(G)$ is reduced weighted Laplacian. Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$ $\hat{D}(G) = \operatorname{diag}(\operatorname{deg} v_1, \ldots, \operatorname{deg} v_n)$ $\hat{\deg v_i} = \sum_{v_i v_i \in E} x_i x_j$ $\hat{A}(G) = adjacency matrix$ (entry $x_i x_i$ for edge $v_i v_i$) Defn 2: $\hat{L}(G) = \partial(G)B(G)\partial(G)^T$ $\partial(G) =$ incidence matrix B(G) diagonal, indexed by edges, entry $\pm x_i x_i$ for edge $v_i v_i$

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Counting trees Matrix-tree theorem

Example (
$$\langle 42, 23 \rangle$$
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$$\hat{L}_r = \begin{pmatrix} 2(1+2+3) & 0 & 0 & -21 & -22 & -23 \\ 0 & 3(1+2) & 0 & -31 & -32 & 0 \\ 0 & 0 & 4(1+2) & -41 & -42 & 0 \\ -21 & -31 & -41 & 1(1+2+3+4) & 0 & 0 \\ -22 & -32 & -42 & 0 & 2(1+2+3+4) & 0 \\ -23 & 0 & 0 & 0 & 0 & 3(1+2) \end{pmatrix}$$
$$\det \hat{L}_r = (1234)(123)(1+2+3+4)(1+2)(1+2+3)(1+2)^2$$

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Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete *d*-dimensional simplicial complex on *n* vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0.
$$\Upsilon_{(d-1)} = K_n^{d-1}$$
 ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- ▶ If 0. holds, then any two of 1., 2., 3. together imply the third.
- When d = 1, coincides with usual definition.

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Example

 $n = 5, d = 2 : \Upsilon = \{123, 124, 125, 134, 135, 245\}$

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C<mark>omplete skeleton</mark> Arbitrary complexes

 $= n^{\binom{n-2}{d}}$

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Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^d)}$$

Counting simplicial spanning trees of K_n^d

Theorem [Kalai '83]

$$\tau(K_n^d) = \sum_{\Upsilon \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}$$

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Weighted simplicial spanning trees of K_n^d

As before,

wt
$$\Upsilon = \prod_{F \in \Upsilon}$$
 wt $F = \prod_{F \in \Upsilon} (\prod_{v \in F} x_v)$

Example

$$\begin{split} & \Upsilon = \{123, 124, 125, 134, 135, 245\} \\ & \text{wt} \ \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3 \end{split}$$

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Weighted simplicial spanning trees of K_n^d

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Example

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Theorem (Kalai, '83)

$$\hat{\tau}(K_n^d) := \sum_{T \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 (\text{wt}\,\Upsilon)$$

$$= (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}$$

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Simplicial spanning trees of arbitrary simplicial complexes

Let Δ be a *d*-dimensional simplicial complex. $\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0.
$$\Upsilon_{(d-1)} = \Delta_{(d-1)}$$
 ("spanning");

1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");

2.
$$\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$$
 ("acyclic");

3.
$$f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$$
 ("count").

- ▶ If 0. holds, then any two of 1., 2., 3. together imply the third.
- When d = 1, coincides with usual definition.

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Theorem (D.-Klivans-Martin, '09)

$$\hat{ au}(\Delta) = rac{| ilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{| ilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_{\Gamma},$$

where

►
$$\Gamma \in SST(\Delta_{(d-1)})$$

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▶ ∂_{Γ} = restriction of ∂_d to faces not in Γ

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- ► $\Gamma \in SST(\Delta_{(d-1)})$
- ▶ ∂_{Γ} = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_{\Gamma} = \partial_{\Gamma} \partial^{T}_{\Gamma}$

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- $\partial_{\Gamma} = restriction of \partial_d$ to faces not in Γ
- reduced Laplacian $L_{\Gamma} = \partial_{\Gamma} \partial^{T}_{\Gamma}$
- Weighted version: Multiply column F of ∂ by x_F

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Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = rac{| ilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{| ilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_{\Gamma},$$

where

- ► $\Gamma \in SST(\Delta_{(d-1)})$
- $\partial_{\Gamma} = restriction of \partial_d$ to faces not in Γ
- reduced Laplacian $L_{\Gamma} = \partial_{\Gamma} \partial^{T}_{\Gamma}$
- Weighted version: Multiply column F of ∂ by x_F

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

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Example: Octahedron

- Vertices 1, 2, 1, 2, 1, 2.
- ▶ Facets 111, 112, 121, 122, 211, 212, 221, 222,
- Γ = 11, 12, 11, 12, 22 spanning tree of 1-skeleton, so remove (rows and columns corresponding to) those edges from weighted Laplacian.
- det $\hat{L}_{\Gamma} = (121212)^3(1+2)(1+2)(1+2)$.

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Color-shifted complexes

Definition (Babson-Novik, '96)

A color-shifted complex is a simplicial complex with:

• vertex set $V_1 \dot{\cup} \dots \dot{\cup} V_r$ (V_i is set of vertices of color i);

$$\blacktriangleright |V_i| = n_i;$$

- every facet contains one vertex of each color; and
- ▶ if v < w are vertices of the same color, then you can always replace w by v.

Note: r = 2 is Ferrers graphs

Example

Octahedron is $\langle 222 \rangle$

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Example $\langle 235, 324, 333 \rangle$

		facets		
1 11	<mark>1</mark> 12	<mark>1</mark> 13	11 4	11 5
1 21	12 2	<mark>1</mark> 23	<mark>1</mark> 24	12 5
<mark>13</mark> 1	13 2	13 3	13 4	13 5
21 1	<mark>2</mark> 12	<mark>21</mark> 3	<mark>21</mark> 4	<mark>2</mark> 15
<mark>22</mark> 1	<mark>22</mark> 2	<mark>2</mark> 23	<mark>2</mark> 24	<mark>2</mark> 25
<mark>23</mark> 1	<mark>23</mark> 2	<mark>23</mark> 3	<mark>23</mark> 4	<mark>23</mark> 5
31 1	3 12	<mark>3</mark> 13	<mark>3</mark> 14	
<mark>3</mark> 21	<mark>3</mark> 22	<mark>3</mark> 23	<mark>3</mark> 24	
<mark>33</mark> 1	<mark>33</mark> 2	<mark>33</mark> 3		

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Example $\langle 235, 324, 333 \rangle$

		facets					r	idges	
1 11	1 12	1 13	1 14	<mark>1</mark> 15	11	<mark>1</mark> 2	1 3		
<mark>1</mark> 21	12 2	<mark>1</mark> 23	<mark>1</mark> 24	<mark>1</mark> 25	21	22	<mark>23</mark>		
1 31	13 2	13 3	1 34	13 5	31	<mark>3</mark> 2	<mark>3</mark> 3		
21 1	<mark>2</mark> 12	<mark>21</mark> 3	<mark>21</mark> 4	<mark>21</mark> 5	11	<mark>1</mark> 2	1 3	<mark>1</mark> 4	1 5
<mark>2</mark> 21	<mark>2</mark> 22	<mark>2</mark> 23	<mark>2</mark> 24	<mark>2</mark> 25	21	22	<mark>2</mark> 3	24	<mark>2</mark> 5
<mark>23</mark> 1	<mark>23</mark> 2	<mark>23</mark> 3	<mark>23</mark> 4	<mark>23</mark> 5	31	<mark>3</mark> 2	<mark>3</mark> 3	<mark>3</mark> 4	
31 1	3 12	3 13	3 14		11	12	13	14	15
3 21	322	3 23	324		21	22	23	24	25
33 1	33 2	33 3	<u> </u>		31	32	33	34	35

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Example $\langle 235, 324, 333 \rangle$

		facets				redu	ced r	idges	
11 1	1 12	1 13	1 14	<mark>1</mark> 15	11	12	13		
1 21	12 2	1 23	<mark>1</mark> 24	<mark>1</mark> 25	21	<mark>2</mark> 2	<mark>23</mark>		
1 31	13 2	13 3	1 34	13 5	31	<mark>3</mark> 2	<mark>3</mark> 3		
011	010	010	014	015	1.1	10	10	14	15
211	<mark>2</mark> 12	<mark>2</mark> 13	<mark>21</mark> 4	21 5	11	12	13	14	15
<mark>2</mark> 21	<mark>2</mark> 22	<mark>2</mark> 23	<mark>2</mark> 24	<mark>2</mark> 25	21	<mark>2</mark> 2	<mark>2</mark> 3	<mark>2</mark> 4	<mark>2</mark> 5
<mark>23</mark> 1	<mark>2</mark> 32	<mark>23</mark> 3	<mark>2</mark> 34	<mark>23</mark> 5	31	<mark>3</mark> 2	<mark>3</mark> 3	<mark>3</mark> 4	
31 1	<mark>3</mark> 12	<mark>3</mark> 13	<mark>3</mark> 14		11	12	13	14	15
<mark>3</mark> 21	<mark>3</mark> 22	<mark>3</mark> 23	<mark>3</mark> 24		21	22	23	24	25
<mark>33</mark> 1	<mark>3</mark> 32	<mark>3</mark> 33			31	3 2	3 3	34	3 5

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Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\begin{aligned} &(1^7 2^7 3^6)(1+2+3)^5(1+2)^3 \\ &\times (1^7 2^6 3^6)(1+2+3)^8(1+2) \\ &\times (1^5 2^5 3^5 4^5 5^4)(1+\dots+5)^2(1+\dots4)(1+\dots3) \end{aligned}$$

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Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$\times (1^{5} 2^{5} 3^{5} 4^{5} 5^{4})(1 + \dots + 5)^{2}(1 + \dots 4)(1 + \dots 3)$

(a)

Definition Enumeration

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

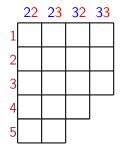
\times (1⁵2⁵3⁵4⁵5⁴)

11	12	13		
21	22	23		
31	<mark>3</mark> 2	<mark>3</mark> 3		
11	12	13	14	15
21	22	23	24	25
31	32	33	34	
11	12	13	14	15
21	22	23	24	25
31	32	33	34	35

(a)

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\times \qquad (1+\cdots+5)^2(1+\cdots4)(1+\cdots3)$$

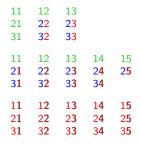


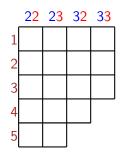
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Definition Enumeration

Enumeration: $\hat{\tau}(\langle 235, 324, 333 \rangle)$

$$\times (1^{5}2^{5}3^{5}4^{5}5^{4})(1 + \dots + 5)^{2}(1 + \dots 4)(1 + \dots 3)$$





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Enumeration

- Theorem (Aalipour-D.)
- When r = 3, this always works.

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Enumeration

Theorem (Aalipour-D.)

When r = 3, this always works.

Conjecture

When r > 3, this always works.

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Enumeration

Theorem (Aalipour-D.)

When r = 3, this always works.

Conjecture

When r > 3, this always works.

Remark

The codimension-1 spanning tree will be a different tree for each color. For each color's factors, treat that color as "last". **Example**: r = 4 (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.

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Proof (via example $\langle 235, 324, 333 \rangle$)

$$det \begin{pmatrix} 22(1+\cdot+5) & 0 & 0 & 0 & \cdots \\ 0 & 23(1+\cdot+5) & 0 & 0 & \cdots \\ 0 & 0 & 22(1+\cdot+4) & 0 & \cdots \\ 0 & 0 & 0 & 33(1+\cdot+3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= (2^2 3^2 2^2 3^2 \cdots) det \begin{pmatrix} 1+\cdot+5 & 0 & 0 & 0 & \cdots \\ 0 & 1+\cdot+5 & 0 & 0 & \cdots \\ 0 & 0 & 1+\cdot+4 & 0 & \cdots \\ 0 & 0 & 0 & 1+\cdot+3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

By "identification of factors" (Martin-Reiner, '03), to show $(1 + \cdot + 5)^2$ is a factor of the det, just show nullspace of this matrix ≥ 2 , when $1 + \cdot + 5 = 0$.

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