Combinatorial Laplacians

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Acknowledgments

Some results part of collaboration with Vic Reiner, and with Carly Klivans and Jeremy Martin

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
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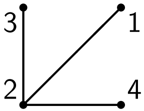
$$\sum_{T \in ST(K_n)} \text{wt}\, T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$

Example: K_4

- 4 trees like: $T =$ 

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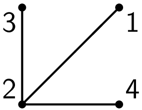
- 4 trees like: $T =$ 

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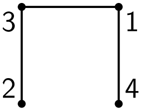
- 12 trees like: $T =$ 

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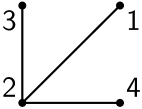
- 4 trees like: $T =$ 

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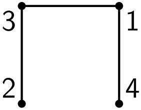
- 12 trees like: $T =$ 

wt $T = (x_1 x_2 x_3 x_4) x_1 x_3$

Total is $(x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2$.

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Definition The Laplacian matrix of graph G, denoted by $L(G)$.

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Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

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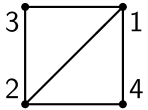
"Reduced": remove rows/columns corresponding to any one vertex

Example



$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Matrix-Tree Theorems

Version I Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then G has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

spanning trees.

Version II G has $|\det L_r(G)|$ spanning trees

Proof [Version II]

$$\det L_r(G) = \det \partial_r(G) \partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$

$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \text{wt}\, T = |\det \hat{L}_r(G)|,$$

where $\hat{L}$ is weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$\hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$\hat{A}(G) =$ adjacency matrix

(entry $x_i x_j$ for edge $v_i v_j$)

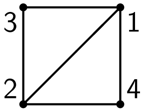
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G) =$ incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example



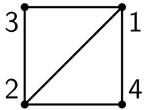
$$\hat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\det \hat{L}_r = (1234)(1+2)(1+2+3+4)$$

Threshold graphs

- Vertices $1, \ldots, n$

Example



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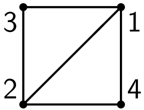

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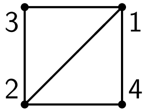
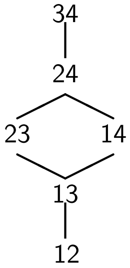
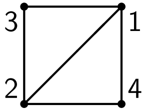
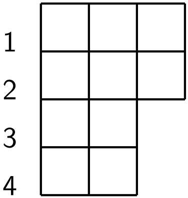




Eigenvalues of threshold graphs

Theorem [Merris '94] Eigenvalues are given by the transpose of the Ferrers diagram of the degree sequence d.





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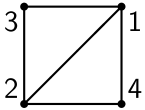

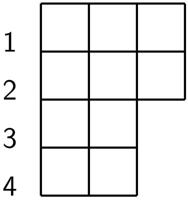

Corollary $\prod_{r \neq 1} (d^T)_r$ spanning trees

Grone-Merris Conjecture

Conjecture (Grone-Merris '94)

$$\begin{aligned} s &\trianglelefteq d^T \\ \lambda_1 &\leq (d^T)_1 = n \\ \lambda_1 + \lambda_2 &\leq (d^T)_1 + (d^T)_2 \\ &\vdots \end{aligned}$$

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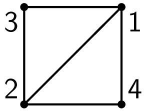
- whole conjecture holds for trees [Stephen '07]

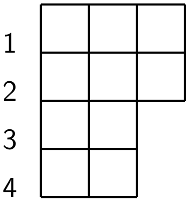
Weighted spanning trees of threshold graphs

Theorem [Martin-Reiner '03; implied by Remmel-Williamson '02]:
If G is threshold, then

$$\sum_{T \in ST(G)} \text{wt } T = (x_1 \cdots x_n) \prod_{r \neq 1} \left(\sum_{i=1}^{(d^T)_r} x_i \right).$$

Example





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coboundary matrix ${\partial_k}^T : \Sigma_{k-1} \to \Sigma_k$

Example: Boundary of tetrahedron

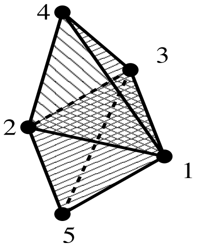
$\partial^T =$

	12	13	14	23	24	34
123	-1	1	0	-1	0	0
124	-1	0	1	0	-1	0
134	0	-1	1	0	0	-1
234	0	0	0	-1	1	-1

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Shifted complexes

- Vertices $1, \ldots, n$



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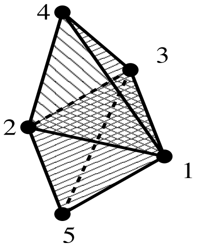

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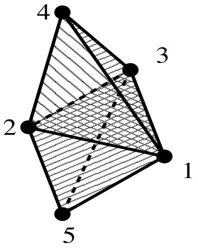
- Equivalently, the k-faces form an initial ideal in the componentwise partial order.



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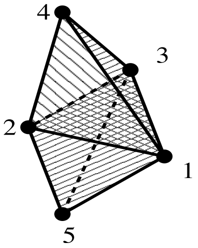
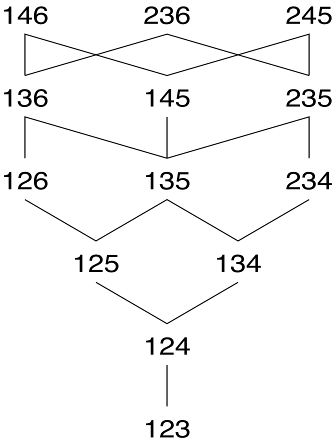
- **Example** (bipyramid with equator) $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Hasse diagram



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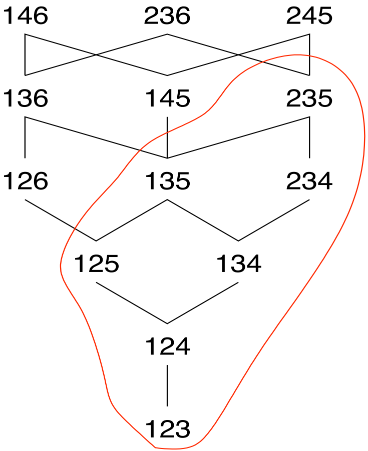
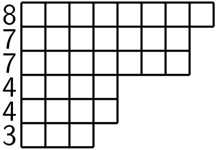


Eigenvalues

Definition d_i is the i-**dimensional degree sequence**
$(d_i)_j = \#$ i-faces containing vertex j.

Example

123	134	234
124	135	235
125	136	236
126	145	



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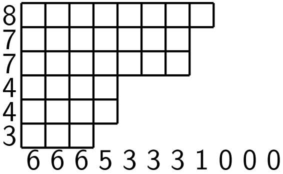

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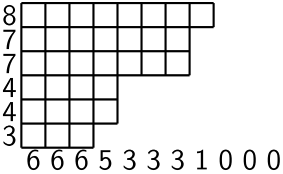

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Open Question

Grone-Merris for simplicial complexes??

Matroids

- ground set $E = \{1, \ldots, n\}$
- Bases $\mathcal{B}$: collection of k-subsets of E satisfying: $\forall B \in \mathcal{B},\ \forall b \in B, \forall B' \in \mathcal{B},\ \exists b' \in B'$ such that

$$(B - b) \cup b' \in \mathcal{B}.$$

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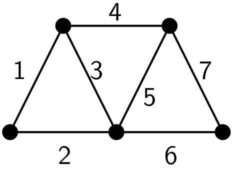
Example (graphical matroid: bases are spanning trees)



$\mathcal{B} =$

1346	2346	1246	1456	1467
1347	2347	1247	1457	2467
1356	2356	1256	2456	
1357	2357	1257	2457	
1367	2367	1267		

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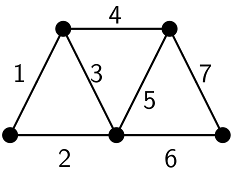

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The simplicial complex formed by taking all subsets of every base $B \in \mathcal{B}$ is the set of independent sets $\mathrm{IN}(M)$ of matroid M.

Eigenvalues of Matroids

For matroids, eigenvalues are more easily described in terms of natural generating function:

$$S_M(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\mathrm{IN}(M)))} q^\lambda$$

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Theorem [Kook-Reiner-Stanton '00]: For a matroid M with ground set E,

$$S_M(t, q) = q^{|E|} \sum_{I \in \mathrm{IN}(M)} t^{\mathrm{rank}(\bar{I})} (q^{-1})^{|\bar{\pi}(I)|},$$

where $\bar{\pi}(I)$ is a function of I involving internal/external activity. In particular, the eigenvalues of M are integers.

Spectral recursion for matroids...

Tutte polynomial deletion-contraction recursion:

$$T_M = T_{M-e} + T_{M/e}$$

$$\mathcal{B}(M-e) = \{B \in \mathcal{B} \colon e \notin B\} \qquad (r = r(M))$$

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Theorem [D '05]: This is true, *i.e.*,

$$S_M = qS_{M-e} + qtS_{M/e} + (1-q)S_{(M-e, M/e)}.$$

. . . and for shifted complexes

Generalize deletion and contraction to arbitrary simplicial complex Δ.

$$\Delta - e = \{F \in \Delta \colon e \notin F\}$$
$$\Delta / e = \{F - e \colon F \in \Delta,\ e \in F\} \qquad = \mathrm{lk}_\Delta\, e$$

$$S_\Delta(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Delta))} q^\lambda$$

Theorem [D '05]: Spectral recursion holds for shifted complexes Δ:

$$S_\Delta = qS_{\Delta - e} + qtS_{\Delta/e} + (1-q)S_{(\Delta - e, \Delta/e)}.$$

A common generalization of shifted complexes and matroids?

Open Question

What is the common generalization of shifted complexes and matroid independence complexes:

- integral Laplacian eigenvalues
- satisfying "spectral recursion"

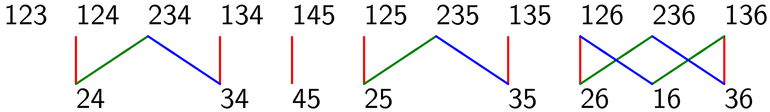
Note that proofs of spectral recursion are completely different for matroids and shifted complexes.

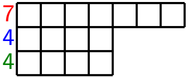
Eigenvalues of shifted pairs

Definition Fix k. Let $\Delta' \subseteq \Delta$ be simplicial complexes. Then

$$d_j(\Delta_k, \Delta'_{k-1}) = \{F \in \Delta_k \colon F - j \notin \Delta'_{k-1}\},$$

and $d(\Delta_k, \Delta'_{k-1}) = (d_1, \dots, d_n)$.





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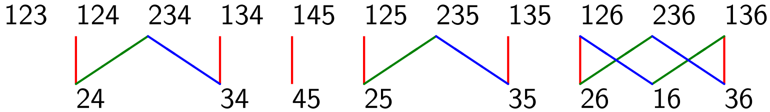

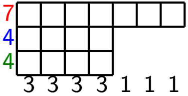

Theorem [D '05]: Eigenvalues of $(\Delta_k, \Delta'_{k-1})$ equal $d(\Delta_k, \Delta'_{k-1})^T$.

Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;

$$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma.$$

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Complete skeleton The k-dimensional complete complex on n vertices, *i.e.*,

$$K_n^k = \{F \subseteq V \colon |F| \leq k+1\}$$

(so $K_n = K_n^1$).

Simplicial spanning trees of K_n^k [Kalai, '83]

$\Upsilon \subseteq K_n^k$ is a **simplicial spanning tree** of K_n^k when:

0. $\Upsilon_{(k-1)} = K_n^{k-1}$ ("spanning");
1. $\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_k(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{k}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $k = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^k

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^k)} \qquad = n^{\binom{n-2}{k}}$$

Counting simplicial spanning trees of K_n^k

Theorem [Kalai '83]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{\binom{n-2}{k}}$$

Weighted simplicial spanning trees of K_n^k

As before,

$$\operatorname{wt}\Upsilon = \prod_{F\in\Upsilon} \operatorname{wt} F = \prod_{F\in\Upsilon} \Big(\prod_{v\in F} x_v\Big)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

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Theorem [Kalai, '83]

$$\sum_{\Upsilon \in SST(K_n)} |\tilde{H}_{k-1}(\Upsilon)|^2 (\operatorname{wt} \Upsilon) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

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(Adin ('92) did something similar for complete r-partite complexes.)

Proof

Proof uses determinant of reduced Laplacian of K_n^k. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial \colon \Delta_k \to \Delta_{k-1}$ boundary

$\partial^T \colon \Delta_{k-1} \to \Delta_k$ coboundary

Weighted version: Multiply column F of ∂ by x_F

Simplicial spanning trees of arbitrary simplicial complexes

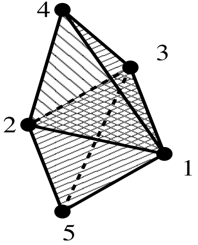
Let Σ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Sigma$ is a **simplicial spanning tree** of Σ when:

0. $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Sigma) - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



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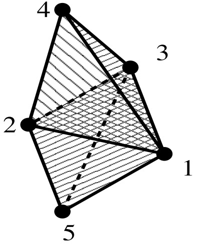

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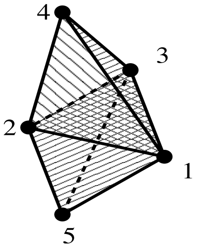

- $3 + 3$ SST's not containing face 123
- 3×3 SST's containing face 123

Total is $(x_1x_2x_3)^3(x_4x_5)^2(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)$.

Simplicial Matrix-Tree Theorem — Version I

- Σ a d-dimensional "metaconnected" simplicial complex
- $(d-1)$-dimensional **(up-down) Laplacian** $L_{d-1} = \partial_{d-1}\partial_{d-1}^T$
- s_d = product of nonzero eigenvalues of L_{d-1}.

Theorem [DKM '09]

$$h_d := \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

Simplicial Matrix-Tree Theorem — Version II

- $\Gamma \in SST(\Sigma_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [DKM '09]

$$h_d = \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Sigma;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Weighted Simplicial Matrix-Tree Theorems

- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply column F of (usual) ∂ by x_F
- $\boldsymbol{\partial}_\Gamma$ = restriction of $\boldsymbol{\partial}_d$ to faces not in Γ
- Weighted reduced Laplacian $\mathbf{L}_\Gamma = \boldsymbol{\partial}_\Gamma \boldsymbol{\partial}_\Gamma^*$

Theorem [DKM '09]

$$\mathbf{h}_d := \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 \prod_{F \in \Upsilon} x_F^2 = \frac{\mathbf{s}_d}{\mathbf{h}_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

$$\mathbf{h}_d = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det \mathbf{L}_\Gamma.$$

A Very fine weighting

Example $F = 235$, $x_F = x_{12}x_{23}x_{35}$

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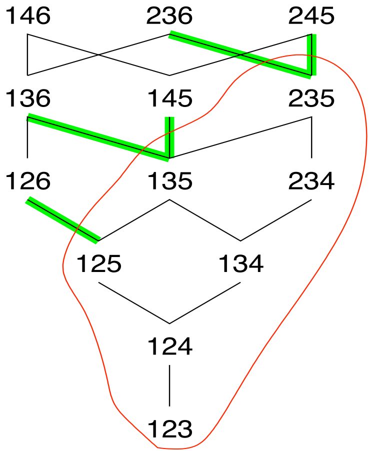
contributes an eigenvalue whose coarse weighting is i.

(Fun exercise is to convince yourself this does match transpose of degree sequence in coarse weighting.)

Critical pairs



What about matroids?

Weighted spanning tree enumerators for independence complexes of matroids seem to factor nicely, but not even a conjectured formula yet.

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Examples:

- $\{124, 134, 234, 125, 135, 235\}$:

$$(x_1x_2x_3x_4x_5)^2(x_1x_2x_3)(x_1+x_2+x_3)(x_4+x_5)$$

- $\{124, 125, 134, 135, 145, 234, 235, 245\}$:

$$(x_1x_2x_3x_4x_5)^3(x_1+x_2+x_3+x_4+x_5)(x_1+x_2)(x_4+x_5)$$

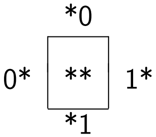
Cubical Complexes

Faces of Q_n, n-dimensional cube: $(0, 1, *)$-strings of length n. Dimension is number of *'s.

Vertices: $(0, 1)$-strings of length n

Edge in direction i: single * in position i.

Boundary: faces with one * converted to 0 or 1.



Cubical Complex: Subset of faces of Q_n such that if a face is included, then so is its boundary.

Spanning Trees

Let $\mathcal{Q}$ be a d-dimensional cubical complex.
$\Upsilon \subseteq \mathcal{Q}$ is a **cubical spanning tree** of $\mathcal{Q}$ when:

0. $\Upsilon_{(d-1)} = \mathcal{Q}_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
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- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
- Works more generally for cellular complexes.

Example

The cubical biprism with equator, the boundary of $\langle$ ***0, **0* $\rangle$

This is the part where you look at the pretty ZomeTool model

- Let's count the spanning trees.

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- $5 + 5$ spanning trees not containing face **00
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- 35 spanning trees total

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Definition The Laplacian matrix of d-dimensional cubical complex $\mathcal{Q}$, denoted by $L(\mathcal{Q})$.

$L(\mathcal{Q}) = \partial(\mathcal{Q})\partial(\mathcal{Q})^T$

$\partial(\mathcal{Q})$ = signed boundary matrix

Example biprism

	00*0	01*0	0*00	0*10	10*0	00*0	11*0	...
0**0								
1**0								
*0*0								
*1*0								
**00								
**10								
...								

Laplacians

Definition The reduced Laplacian matrix of d-dimensional cubical complex $\mathcal{Q}$, denoted by $L_r(\mathcal{Q})$.

$L(\mathcal{Q}) = \partial(\mathcal{Q})\partial(\mathcal{Q})^T$

$$\partial(\mathcal{Q}) = \text{signed boundary matrix}$$

"Reduced": remove rows/columns corresponding to spanning tree of $(d-1)$-dimensional faces

Example biprism

	00*0	01*0	0*00	0*10	10*0	00*0	11*0	...
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1**0								
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Cubical Matrix-Tree Theorem — Version I

Theorem [DKM] If $\mathcal{Q}$ a d-dimensional "metaconnected" cubical complex;
$(d-1)$-dimensional Laplacian $L_{d-1} = \partial_{d-1}\partial_{d-1}{}^T$;
s_d = product of nonzero eigenvalues of L_{d-1}, then

$$h_d := \sum_{\Upsilon \in CST(\mathcal{Q})} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\mathcal{Q})|^2$$

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Example Biprism: $h_d = \frac{(7^2 \cdot 5^4 \cdot 4 \cdot 3^2)(12)}{(7 \cdot 5^3 \cdot 4 \cdot 3^3 \cdot 2^2 \cdot 1)} = 35$

Cubical Matrix-Tree Theorem — Version II

- $\Gamma \in CST(\mathcal{Q}_{(d-1)})$
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Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Skeleta of cubes

Theorem The non-0 eigenvalues of the k-skeleton of Q_n are $2i$ with multiplicity $\binom{n}{i} \times \binom{i-1}{k-1}$ for $i = k, \ldots, n$
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Example 4-cube

k	eigenvalues
4	8^1
3	$8^3 6^4$
2	$8^3 6^8 4^6$
1	$8^1 6^4 4^6 2^4$
(0	2^4)

Weighted tree enumeration on skeleta of cubes

Conjecture

This weighted enumeration has a nice formula.

Example Spanning trees of 2-skeleton of 4-cube, with appropriate weighting:

$$p(123)p(124)p(134)p(234)p(1234)^2$$

where, for instance,

$$p(123) = x_1x_2x_3y_1y_2y_3\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}\right)$$

Shifted cubical complexes

Motivated by shifted simplicial complexes.

Given $\sigma \in Q_n = \{0, 1, *\}^n$, let $dir(\sigma) = \{i \colon \sigma_i = *\}$

A cubical complex $\mathcal{Q} \subseteq \{0, 1, *\}^n$ on n directions is **shifted** if:

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Example

***00	***01	***10	***11
**0*0	**0*1	**1*0	**1*1
**00*	**01*	**10*	**11*
*0**0	*0**1	*1**0	*1**1
*0*0*	*0*1*	*1*0*	*1*1*
0***0	0***1	1***0	1***1
0**0*	0**1*	1**0*	1**1*

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- But the only formula we have is recursive (in terms of deletion and link).

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- Eran Nevo recently found a nice closed formula for homotopy type, which gives the 0 eigenvalues.

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Open Question

Is there a nice closed formula, perhaps involving some new interpretation of degree sequence?

- Eran Nevo recently found a nice closed formula for homotopy type, which gives the 0 eigenvalues.
- One possible strategy, then: Use the 0's and recursion to concoct a formula.

Extremality of shifted cubical complexes

Open Question

Shifted simplicial complexes are extremal in several ways (including f-vectors, algebraic shifting). Are shifted cubical complexes extremal in any way?