The Critical group of a simplicial complex

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April 24, 2010

Counting spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

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$$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$$

$$A(G) = \text{adjacency matrix}$$

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Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) =$ incidence matrix (boundary matrix)

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Matrix-Tree Thm [Kirchhoff] G has $|\det L_r(G)|$ spanning trees.

Definition The reduced Laplacian matrix of G, denoted by $L_r(G)$.

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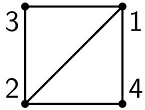
"Reduced": remove rows/columns corresponding to any one vertex

Example



$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

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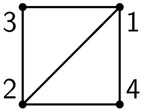

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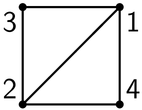

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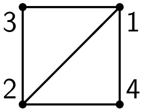

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$$L_r = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$\det L_r = 8$, and there are 8 spanning trees of this graph

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n{-}1 \times n{-}1)$$

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$$\det L_r = \prod \text{eigenvalues}$$
$$= (n - (n-1))(n-0)^{(n-1)-1}$$
$$= n^{n-2}$$

Complete skeleta of simplicial complexes

Simplicial complex $\Delta \subseteq 2^V$;

$$F \subseteq G \in \Delta \Rightarrow F \in \Delta.$$

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Complete skeleton The k-dimensional complete complex on n vertices, *i.e.*,

$$K_n^k = \{F \subseteq V \colon |F| \leq k+1\}$$

(so $K_n = K_n^1$).

Simplicial spanning trees of K_n^k [Kalai, '83]

$\Upsilon \subseteq K_n^k$ is a **simplicial spanning tree** of K_n^k when:

0. $\Upsilon_{(k-1)} = K_n^{k-1}$ ("spanning");
1. $\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_k(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{k}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $k = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^k

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in \mathscr{T}(K_n^k)} \qquad = n^{\binom{n-2}{k}}$$

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Theorem [Kalai '83]

$$\sum_{\Upsilon \in \mathscr{T}(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{\binom{n-2}{k}}$$

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Theorem [Kalai '83]

$$\sum_{\Upsilon \in \mathscr{T}(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{\binom{n-2}{k}}$$

Proof uses determinant of reduced Laplacian of K_n^k. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial \colon \Delta_k \to \Delta_{k-1}$ boundary

$\partial^T \colon \Delta_{k-1} \to \Delta_k$ coboundary

Example $n = 4, k = 2$

$\partial^T =$

	12	13	14	23	24	34
123	-1	1	0	-1	0	0
124	-1	0	1	0	-1	0
134	0	-1	1	0	0	-1
234	0	0	0	-1	1	-1

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Simplicial spanning trees of arbitrary simplicial complexes

Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
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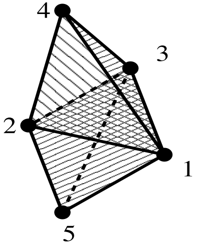
Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Let's figure out all its simplicial spanning trees.

Acyclic in Positive Codimension (APC)

- Denote by $\mathscr{T}(\Delta)$ the set of simplicial spanning trees of Δ.
- **Proposition** $\mathscr{T}(\Delta) \neq \emptyset$ iff Δ is **APC**, *i.e.* (equivalently)
  - homology type of wedge of spheres;
  - $\tilde{H}_j(\Delta; \mathbb{Z})$ is finite for all $j < \dim \Delta$.
- Many interesting complexes are APC.

Simplicial Matrix-Tree Theorem

- Δ a d-dimensional APC complex
- $\Gamma \in \mathscr{T}(\Delta_{(d-1)})$
- ∂_{Γ} = restriction of ∂_d to faces not in Γ
- reduced (up-down) $(d-1)$-dimensional Laplacian $L_\Gamma = \partial_{\Gamma} \partial^*_{\Gamma}$

Theorem [DKM '09]

$$h_d = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

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$L_\Gamma =$

	23	24	25	34	35
23	3	-1	-1	1	1
24	-1	2	0	-1	0
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$\det L_\Gamma = 15.$

Sandpiles and Chip-Firing

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Toppling If $c_i \geq d_i$, then v_i may fire by sending one chip to each of its neighbors.

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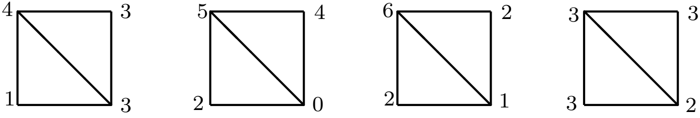

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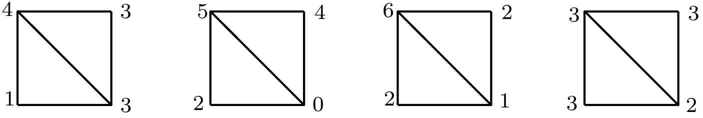

Laplacian Firing v_i is subtracting Lv_i from $(c_1, \ldots, c_n)$.

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- In other words,

$$c \in \ker \partial \subseteq \mathbb{Z}^n$$

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- In other words, instead of $\ker \partial$, we look at

$$K(G) := \ker \partial / \operatorname{im} L$$

the critical group. (It is a graph invariant.)

Spanning trees

Theorem (Biggs '99)

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Corollary

$|K(G)|$ is the number of spanning trees of G.

(Many other proofs.)

Generalize to simplicial complexes

Let Δ be a d-dimensional simplicial complex.

$$C_d(\Delta;\mathbb{Z}) \underset{\partial_d}{\overset{\partial_d^*}{\leftrightarrows}} C_{d-1}(\Delta;\mathbb{Z}) \xrightarrow{\partial_{d-1}} C_{d-2}(\Delta;\mathbb{Z}) \to \cdots$$

$$C_{d-1}(\Delta;\mathbb{Z}) \xrightarrow{L_{d-1}} C_{d-1}(\Delta;\mathbb{Z}) \xrightarrow{\partial_{d-1}} C_{d-2}(\Delta;\mathbb{Z}) \to \cdots$$

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Define

$$K(\Delta) := \ker \partial_{d-1} / \operatorname{im} L_{d-1}$$

where $L_{d-1} = \partial_d \partial_d^*$ is the $(d-1)$-dimensional up-down Laplacian.

What does it look like?

$$K(\Delta) := \ker \partial_{d-1} / \operatorname{im} L_{d-1} \subseteq \mathbb{Z}^m$$

- Put integers on $(d-1)$-faces of Δ. Orient faces arbitrarily. $d = 2$: flow; $d = 3$: circulation; etc.

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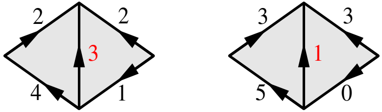
- $d=2$: conservative flow (material does not accumulate or deplete at any vertex); $d=3$: face circulation at each edge adds to zero
- Toppling/firing moves the flow/circulation/whatever to "neighboring" $(d-1)$-faces, across d-faces.



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Simplicial complexes

- To count spanning trees, remove a $(d-1)$-dimensional spanning tree from up-down Laplacian.
- To compute critical group, remove a $(d-1)$-dimensional spanning tree from up-down Laplacian.

Spanning trees

Theorem (DKM)

$$K(\Delta) := (\ker \partial)/(\operatorname{im} L) \cong \mathbb{Z}^r / L_\Gamma$$

where Γ *is a torsion-free* $(d-1)$*-dimensional spanning tree and* $r = \dim L_\Gamma$.

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Corollary

$|K(\Delta)|$ *is the torsion-weighted number of* d*-dimensional spanning trees of* Δ.