Simplicial spanning trees

Art Duval[1] Caroline Klivans[2] Jeremy Martin[3]

[1]University of Texas at El Paso

[2]University of Chicago

[3]University of Kansas

Discrete Mathematics & Representation Theory Seminar
University of California, Davis
May 23, 2008

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

Weighting

vertices? Silly ($n^{n-2}(x_1 \cdots x_n)$)

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

Weighting

vertices? Silly ($n^{n-2}(x_1 \cdots x_n)$)

edges? No nice structure (can't see n^{n-2})

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

Weighting

vertices? Silly ($n^{n-2}(x_1 \cdots x_n)$)

edges? No nice structure (can't see n^{n-2})

both! $\text{wt}\, T = \prod_{e \in T} \text{wt}\, e = \prod_{e \in T}(\prod_{v \in e} x_v)$ Prüfer coding

Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

Weighting

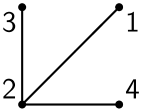
vertices? Silly ($n^{n-2}(x_1 \cdots x_n)$)

edges? No nice structure (can't see n^{n-2})

both! wt $T = \prod_{e \in T}$ wt $e = \prod_{e \in T}(\prod_{v \in e} x_v)$ Prüfer coding

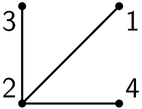
$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$

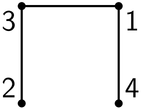
Example: K_4

- 4 trees like: $T =$ 

wt $T = (x_1 x_2 x_3 x_4) x_2^2$

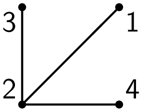
Example: K_4

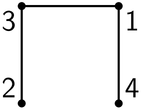
- 4 trees like: $T =$ 

$\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2$
- 12 trees like: $T =$ 

$\text{wt } T = (x_1 x_2 x_3 x_4) x_1 x_3$

Example: K_4

- 4 trees like: $T =$ 

wt $T = (x_1x_2x_3x_4)x_2^2$
- 12 trees like: $T =$ 

wt $T = (x_1x_2x_3x_4)x_1x_3$

Total is $(x_1x_2x_3x_4)(x_1 + x_2 + x_3 + x_4)^2$.

Laplacian

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

Laplacian

Definition The Laplacian matrix of graph G, denoted by $L\ (G)$.

Defn 1: $L(G) = D(G) - A(G)$

$$D(G) = \mathrm{diag}(\deg v_1, \ldots, \deg v_n)$$
$$A(G) = \text{adjacency matrix}$$

Laplacian

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$

$A(G) = \text{adjacency matrix}$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$

Laplacian

Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$$

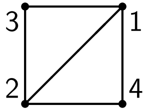
$$A(G) = \text{adjacency matrix}$$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

"Reduced": remove rows/columns corresponding to any one vertex

Example



$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Matrix-Tree Theorems

Version I Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then G has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

spanning trees.

Version II G has $|\det L_r(G)|$ spanning trees

Proof [Version II]

$$\det L_r(G) = \det \partial_r(G) \partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$
$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n-1 \times n-1)$$

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n-1 \times n-1)$$

Version I: Eigenvalues of L are $n - n$ (multiplicity 1), $n - 0$ (multiplicity $n-1$), so

$$\frac{n^{n-1}}{n} = n^{n-2}$$

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n-1 \times n-1)$$

Version I: Eigenvalues of L are $n - n$ (multiplicity 1), $n - 0$ (multiplicity $n-1$), so

$$\frac{n^{n-1}}{n} = n^{n-2}$$

Version II:

$$\begin{aligned} \det L_r &= \prod \text{eigenvalues} \\ &= (n-0)^{(n-1)-1}(n-(n-1)) \\ &= n^{n-2} \end{aligned}$$

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \text{wt } T = |\det \hat{L}_r(G)|,$$

where $\hat{L}$ is weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$\hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$\hat{A}(G) =$ adjacency matrix

(entry $x_i x_j$ for edge $v_i v_j$)

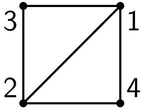
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G) =$ incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example



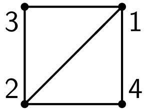
$$\hat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\det \hat{L}_r = (1234)(1+2)(1+2+3+4)$$

Threshold graphs: Order ideal definition

- Vertices $1, \ldots, n$

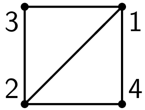
Example



Threshold graphs: Order ideal definition

- Vertices $1, \ldots, n$
- $E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}.$

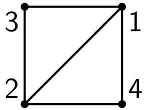
Example

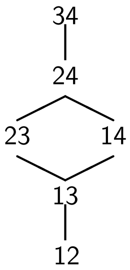


Threshold graphs: Order ideal definition

- Vertices $1, \ldots, n$
- $E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}$.
- Equivalently, the edges form an initial ideal in the componentwise partial order.

Example



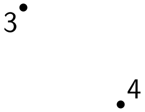


Threshold graphs: Recursive building

Defn 2: Can build recursively, by adding isolated vertices, and coning.

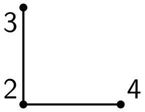
Threshold graphs: Recursive building

Defn 2: Can build recursively, by adding isolated vertices, and coning.



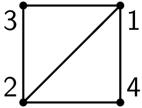
Threshold graphs: Recursive building

Defn 2: Can build recursively, by adding isolated vertices, and coning.



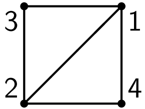
Threshold graphs: Recursive building

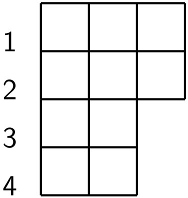
Defn 2: Can build recursively, by adding isolated vertices, and coning.



Eigenvalues of threshold graphs

Theorem [Merris '94] Eigenvalues are given by the transpose of the Ferrers diagram of the degree sequence d.





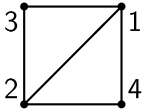
Corollary $\prod_{r \neq 1} (d^T)_r$ spanning trees

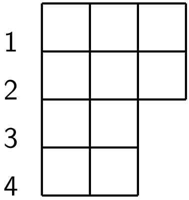
Weighted spanning trees of threshold graphs

Theorem [Martin-Reiner '03; implied by Remmel-Williamson '02]:
If G is threshold, then

$$\sum_{T \in ST(G)} \text{wt}\, T = (x_1 \cdots x_n) \prod_{r \neq 1} \Big(\sum_{i=1}^{(d^T)_r} x_i \Big).$$

Example





$$(1234)(1+2)(1+2+3+4)$$

Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;

$$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma.$$

Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;
$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma$.

Complete skeleton The k-dimensional complete complex on n vertices, *i.e.*,

$$K_n^k = \{F \subseteq V \colon |F| \leq k+1\}$$

(so $K_n = K_n^1$).

Simplicial spanning trees of K_n^k [Kalai, '83]

$\Upsilon \subseteq K_n^k$ is a **simplicial spanning tree** of K_n^k when:

0. $\Upsilon_{(k-1)} = K_n^{k-1}$ ("spanning");
1. $\tilde{H}_{k-1}(\Upsilon;\mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_k(\Upsilon;\mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{k}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $k = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^k

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^k)} \qquad = n^{\binom{n-2}{k}}$$

Counting simplicial spanning trees of K_n^k

Theorem [Kalai '83]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{\binom{n-2}{k}}$$

Weighted simplicial spanning trees of K_n^k

As before,

$$\operatorname{wt} \Upsilon = \prod_{F \in \Upsilon} \operatorname{wt} F = \prod_{F \in \Upsilon} \Big(\prod_{v \in F} x_v \Big)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\operatorname{wt} \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Weighted simplicial spanning trees of K_n^k

As before,

$$\operatorname{wt} \Upsilon = \prod_{F \in \Upsilon} \operatorname{wt} F = \prod_{F \in \Upsilon} \Big(\prod_{v \in F} x_v \Big)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\operatorname{wt} \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Theorem [Kalai, '83]

$$\sum_{T \in SST(K_n)} |\tilde{H}_{k-1}(T)|^2 (\operatorname{wt} T) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

Weighted simplicial spanning trees of K_n^k

As before,

$$\operatorname{wt} \Upsilon = \prod_{F \in \Upsilon} \operatorname{wt} F = \prod_{F \in \Upsilon} \Big(\prod_{v \in F} x_v \Big)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\operatorname{wt} \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Theorem [Kalai, '83]

$$\sum_{T \in SST(K_n)} |\tilde{H}_{k-1}(T)|^2 (\operatorname{wt} T) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

(Adin ('92) did something similar for complete r-partite complexes.)

Proof

Proof uses determinant of reduced Laplacian of K_n^k. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial \colon \Delta_k \to \Delta_{k-1}$ boundary

$\partial^T \colon \Delta_{k-1} \to \Delta_k$ coboundary

Weighted version: Multiply column F of ∂ by x_F

Example $n = 4, k = 2$

$\partial^T =$

	12	13	14	23	24	34
123	-1	1	0	-1	0	0
124	-1	0	1	0	-1	0
134	0	-1	1	0	0	-1
234	0	0	0	-1	1	-1

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Simplicial spanning trees of arbitrary simplicial comlexes

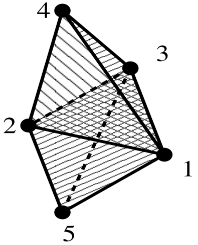
Let Σ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Sigma$ is a **simplicial spanning tree** of Σ when:

0. $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Sigma) - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235\rangle$



Let's figure out all its simplicial spanning trees.

Metaconnectedness

- Denote by $\mathscr{T}(\Sigma)$ the set of simplicial spanning trees of Σ.
- **Proposition** $\mathscr{T}(\Sigma) \neq \emptyset$ iff Σ is **metaconnected**, *i.e.* (equivalently)
  - homology type of wedge of spheres;
  - $\tilde{H}_j(\Sigma; \mathbb{Z})$ is finite for all $j < \dim \Sigma$.
- Many interesting complexes are metaconnected, including everything we'll talk about.

Simplicial Matrix-Tree Theorem — Version I

- Σ a d-dimensional metaconnected simplicial complex
- $(d-1)$-dimensional **(up-down) Laplacian** $L_{d-1} = \partial_{d-1}\partial_{d-1}^T$
- s_d = product of nonzero eigenvalues of L_{d-1}.

Theorem [DKM]

$$h_d := \sum_{\Upsilon \in \mathscr{T}(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

Simplicial Matrix-Tree Theorem — Version II

- $\Gamma \in \mathscr{T}(\Sigma(d-1))$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [DKM]

$$h_d \;=\; \sum_{\Upsilon \in \mathscr{T}(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\Sigma;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Weighted Simplicial Matrix-Tree Theorems

- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply column F of (usual) ∂ by x_F
- $\boldsymbol{\partial}_\Gamma$ = restriction of $\boldsymbol{\partial}_d$ to faces not in Γ
- Weighted reduced Laplacian $\mathbf{L} = \boldsymbol{\partial}_\Gamma \boldsymbol{\partial}_\Gamma^*$

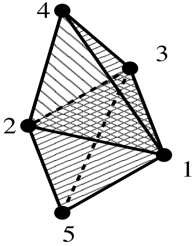
Theorem [DKM]

$$\mathbf{h}_d := \sum_{\Upsilon \in \mathscr{T}(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 \prod_{F \in \Upsilon} x_F^2 = \frac{\mathbf{s}_d}{\mathbf{h}_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

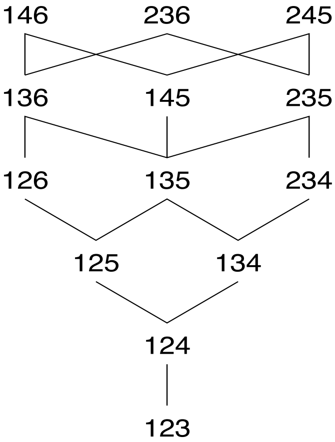
$$\mathbf{h}_d = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det \mathbf{L}_\Gamma.$$

Definition of shifted complexes

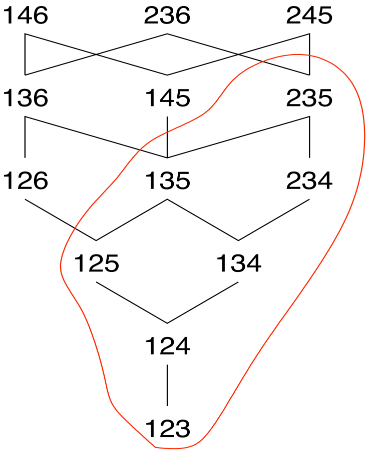
- Vertices $1, \ldots, n$
- $F \in \Sigma, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Sigma$
- Equivalently, the k-faces form an initial ideal in the componentwise partial order.
- **Example** (bipyramid with equator)
  $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Hasse diagram



Hasse diagram



Links and deletions

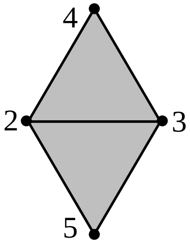
- Deletion, $\operatorname{del}_1 \Sigma = \{G \colon 1 \notin G, G \in \Sigma\}$.
- Link, $\operatorname{lk}_1 \Sigma = \{F - 1 \colon 1 \in F, F \in \Sigma\}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

$$\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$$

Links and deletions

- Deletion, $\operatorname{del}_1 \Sigma = \{G \colon 1 \notin G, G \in \Sigma\}$.
- Link, $\operatorname{lk}_1 \Sigma = \{F - 1 \colon 1 \in F, F \in \Sigma\}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

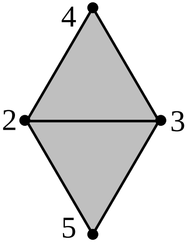
$$\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$$
$$\operatorname{del}_1 \Sigma = \langle 234, 235 \rangle$$

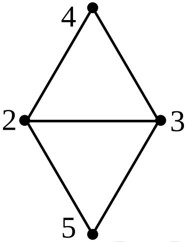


Links and deletions

- Deletion, $\operatorname{del}_1 \Sigma = \{G \colon 1 \notin G, G \in \Sigma\}$.
- Link, $\operatorname{lk}_1 \Sigma = \{F - 1 \colon 1 \in F, F \in \Sigma\}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

$$\begin{aligned} \Sigma &= \langle 123, 124, 125, 134, 135, 234, 235 \rangle \\ \operatorname{del}_1 \Sigma &= \langle 234, 235 \rangle \\ \operatorname{lk}_1 \Sigma &= \langle 23, 24, 25, 34, 35 \rangle \end{aligned}$$





Weighted spanning trees

- In Weighted Simplicial Matrix Theorem II, pick Γ to be the set of all $(d-1)$-dimensional faces containing vertex 1.

Weighted spanning trees

- In Weighted Simplicial Matrix Theorem II, pick Γ to be the set of all $(d-1)$-dimensional faces containing vertex 1.
- $H_{d-2}(\Gamma;\mathbb{Z})$ and $H_{d-2}(\Sigma;\mathbb{Z})$ are trivial, so,

$$\mathbf{h}_d = \det \mathbf{L}_\Gamma$$

Weighted spanning trees

- In Weighted Simplicial Matrix Theorem II, pick Γ to be the set of all $(d-1)$-dimensional faces containing vertex 1.
- $H_{d-2}(\Gamma;\mathbb{Z})$ and $H_{d-2}(\Sigma;\mathbb{Z})$ are trivial, so, by some easy linear algebra,

$$\mathbf{h}_d = \det \mathbf{L}_\Gamma = \Big(\prod_{\sigma\in \mathrm{lk}_1 \Sigma} X_\sigma\Big) \det(X_1 I + \mathbf{L}_{\mathrm{del}_1 \Sigma, d-1})$$

where $X_i{=}x_i^2$

Weighted spanning trees reduce to eigenvalues

- In Weighted Simplicial Matrix Theorem II, pick Γ to be the set of all $(d-1)$-dimensional faces containing vertex 1.
- $H_{d-2}(\Gamma;\mathbb{Z})$ and $H_{d-2}(\Sigma;\mathbb{Z})$ are trivial, so, by some easy linear algebra,

$$\begin{aligned}\mathbf{h}_d = \det \mathbf{L}_\Gamma &= \Big(\prod_{\sigma\in \operatorname{lk}_1 \Sigma} X_\sigma\Big) \det(X_1 I + \mathbf{L}_{\operatorname{del}_1 \Sigma, d-1}) \\ &= \Big(\prod_{\sigma\in \operatorname{lk}_1 \Sigma} X_\sigma\Big)\Big(\prod_{\substack{\lambda \text{ e'val of} \\ \mathbf{L}_{\operatorname{del}_1 \Sigma, d-1}}} X_1 + \lambda\Big),\end{aligned}$$

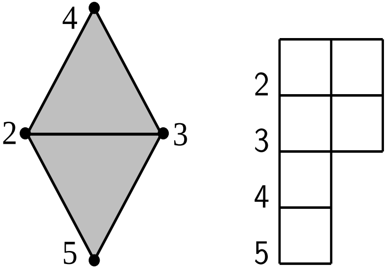
where $X_i = x_i^2$

Eigenvalues

Theorem [D-Reiner, '02]

Non-zero eigenvalues are given by the transpose of the Ferrers diagram of the (generalized) degree sequence d.

Example

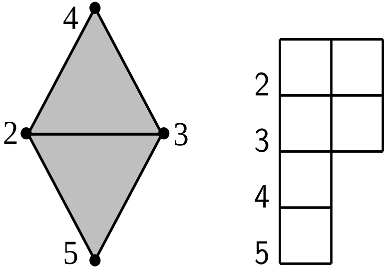


Weighted Eigenvalues

Theorem [DKM]

Non-zero weighted eigenvalues are given by the transpose of the Ferrers diagram of the (generalized) degree sequence d.

Example



$$(2+3)(2+3+4+5)$$

Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \operatorname{lk}_1 \Sigma$, , $\Delta = \operatorname{del}_1 \Sigma$,

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \operatorname{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$

$\Delta = \operatorname{del}_1 \Sigma = \langle 234, 235 \rangle$

Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \mathrm{lk}_1\,\Sigma$, , $\Delta = \mathrm{del}_1\,\Sigma$,

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235\rangle$ again

$$\Lambda = \mathrm{lk}_1\,\Sigma = \langle 23, 24, 25, 34, 35\rangle$$

$$\Delta = \mathrm{del}_1\,\Sigma = \langle 234, 235\rangle$$

$$\mathbf{h}_2 = (23)(24)(25)(34)(35)(1 + (2 + 3))(1 + (2 + 3 + 4 + 5))111$$

Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \mathrm{lk}_1 \Sigma$, , $\Delta = \mathrm{del}_1 \Sigma$,

$$\mathbf{h}_d = \prod_{\sigma \in \Lambda} X_{\sigma \cup 1} \prod_r ((\sum_{i=1}^{1+(d(\Delta)^T)_r} X_i)/X_1)$$

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \mathrm{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$

$\Delta = \mathrm{del}_1 \Sigma = \langle 234, 235 \rangle$

$$\mathbf{h}_2 = (23)(24)(25)(34)(35)(1 + (2 + 3))(1 + (2 + 3 + 4 + 5))111$$
$$= (123)(124)(125)(134)(135)((1 + 2 + 3)/1)((1 + 2 + 3 + 4 + 5)/1)$$

Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \mathrm{lk}_1 \Sigma$, $\tilde{\Lambda} = 1 * \Lambda$, $\Delta = \mathrm{del}_1 \Sigma$, $\tilde{\Delta} = 1 * \Delta$.

$$\mathbf{h}_d = \prod_{\sigma \in \Lambda} X_{\sigma \cup 1} \prod_r \Big(\Big(\sum_{i=1}^{1+(d(\Delta)^T)_r} X_i \Big) / X_1 \Big)$$

$$= \prod_{\sigma \in \tilde{\Lambda}} X_\sigma \prod_r \Big(\Big(\sum_{i=1}^{(d(\tilde{\Delta})^T)_r} X_i \Big) / X_1 \Big).$$

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \mathrm{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$ $\quad \tilde{\Lambda} = \langle 123, 124, 125, 134, 135 \rangle$

$\Delta = \mathrm{del}_1 \Sigma = \langle 234, 235 \rangle$ $\quad \tilde{\Delta} = \langle 1234, 1235 \rangle$

$$\mathbf{h}_2 = (23)(24)(25)(34)(35)(1 + (2+3))(1 + (2+3+4+5))111$$
$$= (123)(124)(125)(134)(135)((1+2+3)/1)((1+2+3+4+5)/1)$$

Fine weighting

- Weight $F = \{i_1 < \cdots < i_k\}$ by

$$x_{1,i_1} x_{2,i_2} \cdots x_{k,i_k}.$$

- Keeps track of where in each face the vertex appears.
- Can generalize our results on tree enumeration and eigenvalues, but things get more complex.

Conjecture for Matroid Complexes

$\mathbf{h}_d$ again seems to factor nicely, though we can't describe it yet.

Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).

Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
- Then we can define boundary map, and all the algebraic topology, including Laplacian.

Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
- Then we can define boundary map, and all the algebraic topology, including Laplacian.
- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).

Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
- Then we can define boundary map, and all the algebraic topology, including Laplacian.
- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).
- Complete skeleta are very nicely behaved for eigenvalues, spanning trees.

Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
- Then we can define boundary map, and all the algebraic topology, including Laplacian.
- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).
- Complete skeleta are very nicely behaved for eigenvalues, spanning trees.
- Cubical analogue of shifted complexes have integer eigenvalues; still working on trees.

Definition of color-shifted complexes

- Set of colors
- n_c vertices, $(c,1),(c,2),\ldots(c,n_c)$ of color c.
- Faces contain at most one vertex of each color.
- Can replace (c,j) by (c,i) in a face if $i<j$.
- Example: Faces written as (red,blue,green): 111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 222, 223, 231,232.

Conjecture for complete color-shifted complexes

Let Δ be the color-shifted complex generated by the face with red a, blue b, green c. Let the red vertices be $x_1, \ldots, x_a$, the blue vertices be $y_1, \ldots, y_b$, and the green vertices be $z_1, \ldots, z_c$.

Conjecture

$$\mathbf{h}_d(\Delta) = (\prod_{i=1}^{a} x_i)^{b+c-1} (\prod_{j=1}^{b} y_j)^{a+c-1} (\prod_{k=1}^{c} z_k)^{a+b-1}$$
$$\times (\sum_{i=1}^{a} x_i)^{(b-1)(c-1)} (\sum_{j=1}^{b} y_j)^{(a-1)(c-1)} (\sum_{k=1}^{c} z_k)^{(a-1)(b-1)}$$

Notes on conjecture

- This is with coarse weighting. Every vertex v has weight x_v, and every face F has weight

$$x_F = \prod_{v \in F} x_v .$$

- The case with two colors is a (complete) **Ferrers graph**, studied by Ehrenborg and van Willigenburg.