The Importance of being Equivalent: The Ubiquity of equivalence relations in mathematics, K-16+

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Equality

What are the important properties of equality?

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Definitions and motivation Examples More theory Fractions Geometry Real life Summary

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Along with operations (such as +),

$$\begin{array}{ccc} \text{if} & a=b\\ and & c=d\\ \hline \text{then} & a+c=b+d \end{array}$$

(substitution)

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Fractions Geometry Real life Summary

One reason fractions are hard

 $\frac{2}{3}+\frac{1}{5}=$

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$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15}$$

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Fractions Geometry Real life Summary

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Fractions Geometry Real life Summary

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Fractions Geometry Real life Summary

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$$\frac{2}{3} + \frac{1}{5} = \\ \frac{10}{15} + \frac{3}{15} = \frac{13}{15} \\ \text{We have to use } \frac{2}{3} = \frac{10}{15} \text{ and } \frac{1}{5} = \frac{3}{15}. \\ \text{Questions:}$$

• If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?

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Fractions Geometry Real life Summary

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- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?

Fractions Geometry Real life Summary

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 We have to use $\frac{2}{3} = \frac{10}{15}$ and $\frac{1}{5} = \frac{3}{15}$. Questions:

- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?
- ► Would we use something else in another situation, or should we always use ¹⁰/₁₅?

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Fractions Geometry Real life Summary

Equivalent fractions

Definition: $\frac{a}{b} \sim \frac{c}{d}$ if they reduce to the same fraction (ad = bc). It's easy to check the following properties:

Fractions Geometry Real life Summary

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Fractions Geometry Real life Summary

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Fractions Geometry Real life Summary

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Because of these three properties, we say \sim is an equivalence relation

Partitioning fractions

Because fraction equivalence is an equivalence relation, we can partition fractions as follows:

 $\frac{a}{b}$ and $\frac{c}{d}$ are in the same part ("equivalence class") if $\frac{a}{b} \sim \frac{c}{d}$.

$\frac{1}{2}$	$\frac{17}{34}$	$\frac{2}{3}$	$\frac{10}{15}$	$\frac{1}{5}$	$\frac{3}{15}$	$\frac{4}{7}$	<u>20</u> 35
$\frac{4}{8}$	$\frac{6}{12}$	$\frac{4}{6}$	$\frac{14}{21}$	$\frac{10}{50}$	$\frac{8}{40}$	$\frac{40}{70}$	$\frac{16}{28}$
$\frac{10}{20}$	$\frac{7}{14}$	$\frac{20}{20}$	$\frac{8}{12}$	$\frac{2}{10}$	$\frac{7}{35}$	$\frac{8}{14}$	<u>36</u> 63

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Rules for partitions:

- Everything is in exactly one part
- No empty part

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Fractions Geometry Real life Summary

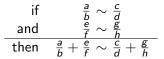
Adding fractions (revisited)

if	$rac{a}{b}\sim rac{c}{d}$
and	$rac{e}{f} \sim rac{g}{h}$

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Adding fractions (revisited)



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Adding fractions (revisited)

$$\begin{array}{ccc} \text{if} & \frac{a}{b} \sim \frac{c}{d} \\ \text{and} & \frac{e}{f} \sim \frac{g}{b} \\ \text{then} & \frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{b} \end{array}$$

So, really we should say

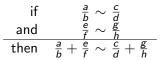
$$\begin{bmatrix} 2\\ 3 \end{bmatrix} + \begin{bmatrix} 1\\ 5 \end{bmatrix} = \begin{bmatrix} 13\\ 15 \end{bmatrix},$$

because anything equivalent to $\frac{2}{3}$ plus anything equivalent to $\frac{1}{5}$ "equals" something equivalent to $\frac{13}{15}$.

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Fractions Geometry Real life Summary

Adding fractions (revisited)



So, really we should say

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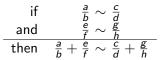
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Fractions Geometry Real life Summary

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- But it's hard to compute unless we pick the right representative.
- In other settings, we stick to the fraction in lowest terms, a distinguished representative.

Fractions Geometry Real life Summary

Similarity, congruence, etc.

Some equivalence relations from geometry:

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Similarity, congruence, etc.

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- Similarity
 - same "shape", possibly different size
 - can get via dilation, reflection, rotation, translation

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- Same shape, size, chirality, orientation
 - can get via translation
- Same shape, size, chirality, orientation, position
 - equality

Fractions Geometry Real life Summary

Finer partitions

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Finer partitions

As we go down that ladder, we refine the partition, by splitting each part into more parts.

Finer partitions

- As we go down that ladder, we refine the partition, by splitting each part into more parts.
- Different situations call for different interpretations of when two shapes are "the same".

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Definitions and motivation Examples More theory Fractions Geometry Real life

Cars

My car is different than yours (not equal), even if they are the same model.

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Cars

- My car is different than yours (not equal), even if they are the same model.
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 - Which Japanese car?

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- But if they are the same model, they have the same maintenance schedule
- But even then, they may not be repaired on the same schedule.
- "A Japanese car needs you to hold the handle when you lock it. but an American car does not."
 - Which Japanese car?
 - Which American car?

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Definitions and motivation Examples More theory Real life Summary

Names

Gwendolen: ... my ideal has always been to love some one of the name of Ernest. There is something in that name that inspires absolute confidence. The moment Algernon first mentioned to me that he had a friend called Ernest, I knew I was destined to love you.

Cecily: ... it had always been a girlish dream of mine to love some one whose name was Ernest. There is something in that name that seems to inspire absolute confidence. I pity any poor married woman whose husband is not called Ernest.

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Definitions and motivation Examples More theory Fractions Geometry Real life

Money

► At the store, 1 dollar equals 4 quarters equals 10 dimes.

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Fractions Geometry Real life Summary

Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.

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- > At parking meters, quarters good, everything else bad.

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- At my vending machine, dollar good, coins bad.
- At parking meters, quarters good, everything else bad.
- Everywhere, pennies bad.

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Fractions Geometry **Real life** Summary

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

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Fractions Geometry Real life Summary

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$? • When it's apple pie.

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Fractions Geometry **Real life** Summary

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids

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Fractions Geometry **Real life** Summary

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids and no knife.

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Fractions Geometry Real life Summary

What have we seen equivalences do?

recognize and categorize

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- recognize and categorize
 - by partition (maybe by distinguished representative)

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Definitions and motivation

Summarv

What have we seen equivalences do?

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Definitions and motivation

Summarv

What have we seen equivalences do?

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Definitions and motivation

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- define functions (have to be well-defined)

Definitions and motivation Examples

Summarv

What have we seen equivalences do?

- recognize and categorize
 - by partition (maybe by distinguished representative)
 - by transitivity and basic equivalences
- respect operations (make operations easier)
- define functions (have to be well-defined)
- refine partitions

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Where else do we see this?

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Where else do we see this?

Glad you asked

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Elementary

Regrouping

To do multidigit addition and subtraction,

 $436 = 400 + 30 + 6 = 400 + 20 + 16 = 300 + 130 + 6 = \cdots$

- Different representations are better or worse for different addition and subtraction problems.
- Using base-10 blocks, these all make different (but "equivalent") pictures.

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"Unique" factorization

Completely factor 60, as

 $2 \times 2 \times 3 \times 5 = 2 \times 3 \times 2 \times 5 = 5 \times 2 \times 2 \times 3 = \cdots$

Natural to say these are all the "same"; once we do, we get unique factorization into primes.

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- Natural to say these are all the "same"; once we do, we get unique factorization into primes.
- Distinguished representative is usually to arrange primes from smallest to largest.
- In context of factorization, 6 × 10 and 4 × 15 are different, even though usually 6 × 10 = 4 × 15.

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0.999 . . .

$0.999\ldots=1$

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0.999	

right?

▶ 0.999... isn't even a number, it's an infinite process

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right?

 0.999... isn't even a number, it's an infinite process that gets arbitrarily close to 1

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right?

- 0.999... isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.

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right?

- 0.999... isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).

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0.999...

$0.999\ldots = 1$

right?

- 0.999... isn't even a number, it's an infinite process that gets arbitrarily close to 1
- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's close enough for everything we do.

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- "gets arbitrarily close to" is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's close enough for everything we do.
- And allowing it (and all its infinite process buddies) allows us to say things like $\sqrt{2}$ and *e* are numbers, on the number line.

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Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

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Algebraic expressions

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right?

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Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

▶ The two expressions are equal for all values of x.

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Algebraic expressions

$$(x-1)(x+1) = x^2 - 1$$

right?

- The two expressions are equal for all values of x.
- Being equal for all values of [all relevant variables] is an equivalence relation.

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Algebraic expressions

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- ▶ But it is not so obvious when expressions are equivalent.

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- ► The two expressions are equal for all values of *x*.
- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it's good enough for everything we do.
- But it is not so obvious when expressions are equivalent.
- There are many different ideas of "distinguished representative".

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Algebraic equations

3x + 7 = 22 is the same as 3x = 15,

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Algebraic equations

3x + 7 = 22 is the same as 3x = 15, right?

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Algebraic equations

- 3x + 7 = 22 is the same as 3x = 15, right?
 - ▶ The two equations have the same solution set for *x*.

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Algebraic equations

- 3x + 7 = 22 is the same as 3x = 15, right?
 - The two equations have the same solution set for x.
 - Having the same solution set for [all relevant variables] is an equivalence relation.

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High school

Algebraic equations

3x + 7 = 22 is the same as 3x = 15, right?

- The two equations have the same solution set for x.
- Having the same solution set for [all relevant variables] is an equivalence relation.
- The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.

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Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$,

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With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).

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- With combinations (like this), order does not matter, so it's an equivalence relation on ordered lists (permutations).
- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.

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Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$, right?

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- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.

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- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
- ▶ Usually, the distinguished representative (ordered list) to represent a combination (unordered list) is to put the items "in order"; for instance: {1, 10, 17, 23, 42, 54}.

More about counting

The relation between counting formulas for permutations and combinations reminds us of one more thing equivalence relations are good for (that doesn't show up in the elementary examples): If the equivalence classes all have the same number of elements (perhaps by some symmetry argument), then

size of set = (number of classes) \times (size of classes)

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- Distinguished representative is often to start at the origin. But to see how to add two vectors, we should move the starting point of the second one.
- This equivalence relation respects vector addition and scalar multiplication.

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Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by m.
- Example: Even and odd (m = 2).

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Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by m.
- Example: Even and odd (m = 2).
- This equivalence relation respects addition and multiplication.
- Example: Last digit arithmetic (m = 10).

Anti-differentiation

Solve

$$f'(x) = 3x^2$$

• "Answer" is
$$x^3 + C$$
.

- This really means the equivalence class of functions that can be written in this form
- The equivalence relation is $f \sim g$ if f g is a constant.
- This equivalence relation respects addition, multiplication by a constant, which is why those are easy to deal with in anti-differentiation.

College

Linear Differential equations Solve

$$y''' - 5y'' + y' - y = 3x^2$$

Solutions of the form

$$y = y_0 + y_p$$

where y_0 is the general solution to the homogeneous equation, and y_p is a particular solution.

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Similarly for the matrix equation

$$Mx = b.$$

Gaussian elimination in matrices

- Consists of a series of elementary row operations that do not change the solution set.
- So at the end, we have a nicer representative of the same equivalence class (of systems with the same solution).

Cardinality

What is the cardinality of a set?

- It's not defined as a function, per se
- We just say when two sets have the same cardinality.
- That's an equivalence relation, not a function.
- ▶ There are some distinguished representatives: 0; 1; 2; ...; ℕ; ℝ.

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Isomorphisms

- ► Graphs, groups, topological spaces, partial orders, etc.
- Two objects are isomorphic if they have the same structure that we care about, even though they may look very different.
- It can be difficult to determine when two objects are isomorphic.

Why do some equivalence relations respect addition?

What we really need is to make sure that [0] acts like the additive identity:

$$[0] + [0] = [0].$$

Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

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Similarly, the nonabelian case gives rise to normal subgroups.

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What we really need is to make sure that [0] acts like the multiplicative "killer":

$$[0] \times [x] = [0]$$

for all [x].

Along with the subgroup condition (for addition), this is just the definition of ideal.

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Why do some equivalence relations respect order?

- What we really need is to make sure positive × positive is positive, positive + positive is positive.
- If these hold, it's easy to check our usual rules about addition and multiplication respecting order.

Well-defined functions

- Functions on equivalence classes are often defined in terms of first picking a representative.
- Operation-preserving is a special case of this.
- We have to make sure it doesn't matter which representative we pick.
- ▶ In the algebraic setting, this usually reduces to checking that f([0]) = 0.

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