Critical Groups of Simplicial Complexes

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Sandpiles and chip-firing

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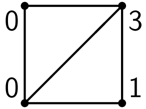

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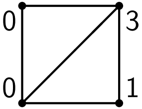

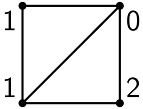

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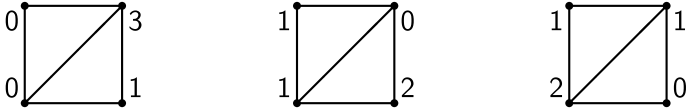

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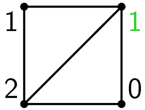

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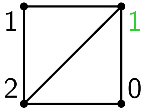

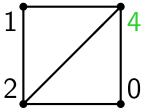

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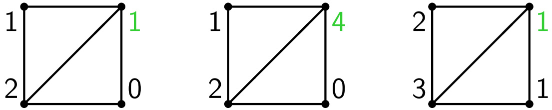

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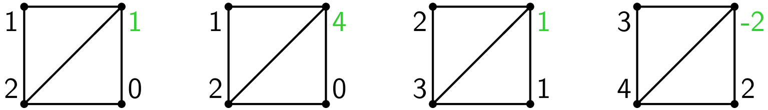

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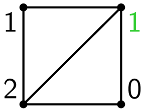

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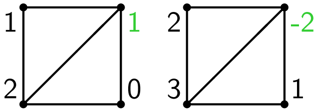

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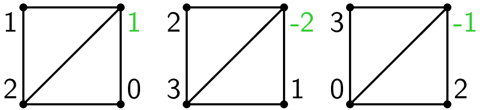

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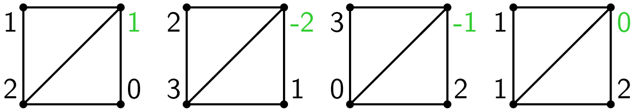

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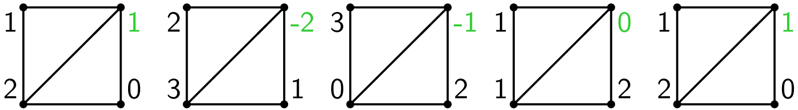
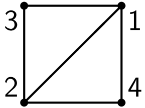


Fact: Every configuration topples to a unique critical configuration.

Laplacian

Let's make a matrix of how chips move when each vertex fires:



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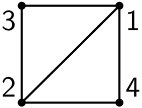

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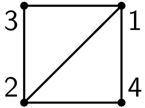

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where

$$\partial_1 = \begin{array}{c|ccccc} & 12 & 13 & 14 & 23 & 24 \\ \hline 1 & -1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & -1 & -1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 \end{array}$$

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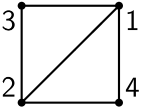

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So firing v is subtracting Lv (row/column v from L) from $(c_1, \ldots, c_n)$.

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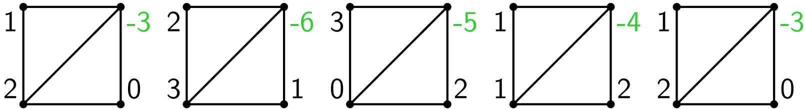

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- Recall every configuration is equivalent to a critical configuration.
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- In other words, instead of $\ker \partial_0$, we look at

$$K(G) := (\ker \partial_0)/(\operatorname{im} L) = (\ker \partial_0)/(\operatorname{im} \partial_1 \partial_1^T)$$

the critical group. (It is a graph invariant.)

Reduced Laplacian and spanning trees

Theorem (Biggs '99)

$$K := (\ker \partial_0)/(\operatorname{im} L) \cong \mathbb{Z}^{n-1}/(\operatorname{im} L_r),$$

where L_r denotes reduced Laplacian; remove row and column corresponding to source vertex.

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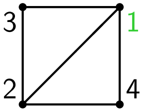
and $|\det L_r|$ counts spanning trees. □

Example



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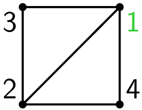

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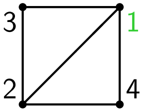

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$\det L_r = 8$, and there are 8 spanning trees of this graph

Generalize to simplicial complexes

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Simplicial spanning trees of arbitrary simplicial complexes

Let Δ be a d-dimensional simplicial complex, and assume it is **APC**, acyclic in positive codimension, i.e., $\tilde{H}_j(\Delta;\mathbb{Z})$ is finite for all $j < d$.

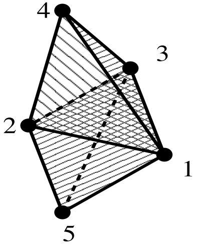
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon;\mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Let's figure out all its simplicial spanning trees.

Reduced Laplacians to count spanning trees

Let $\mathcal{T}(\Delta)$ denote the spanning trees of Δ.

- Δ a d-dimensional APC complex
- $\Gamma \in \mathcal{T}(\Delta_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced (up-down) $(d-1)$-dimensional Laplacian $L_\Gamma = \partial_\Gamma {\partial^T}_\Gamma$

Simplicial Matrix-Tree Theorem [DKM '09]

$$h_d = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

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$\det L_\Gamma = 15.$

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- To compute critical group, remove a $(d-1)$-dimensional spanning tree from up-down Laplacian.

Spanning trees

Theorem (DKM)

$$K(\Delta) := (\ker \partial_{d-1})/(\operatorname{im} L_{d-1}) \cong \mathbb{Z}^t/(\operatorname{im} L_\Gamma)$$

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Corollary

$|K(\Delta)|$ is the torsion-weighted number of d-dimensional spanning trees of Δ.

Proof.

$|K(\Delta)| = |\mathbb{Z}^t/(\operatorname{im} L_\Gamma)| = |\det L_\Gamma|$, which counts (torsion-weighted) spanning trees. □

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- conservative flow

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$$K(\Delta) := (\ker \partial_{d-1})/(\operatorname{im} L_{d-1}) \subseteq \mathbb{Z}^m$$

- Put integers on $(d-1)$-faces of Δ. Orient faces arbitrarily. $d = 2$: flow; $d = 3$: circulation; etc.
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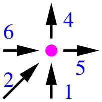
  - $d = 2$: chips do not accumulate or deplete at any vertex;



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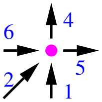

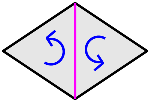

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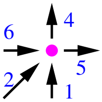
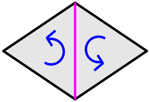
- By theorem, just specify values off the spanning tree.

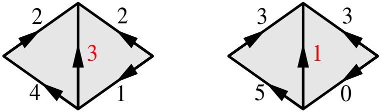




Firing faces

$$K(\Delta) := (\ker \partial_{d-1})/(\mathrm{im}\, L_{d-1}) \subseteq \mathbb{Z}^m$$

Toppling/firing moves the flow to "neighboring" $(d-1)$-faces, across d-faces.



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- But this misses the sense of "critical".
- Main obstacle is idea of what is "positive".

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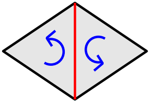

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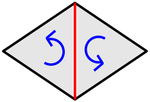

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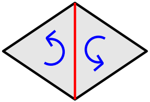

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- So $K(\Delta)$ has a single generator, so it is cyclic.
- $|K(\Delta)|$ is the number of spanning trees, and there is one tree for every facet (remove that facet for the tree).

□

Riemann-Roch Theorem for Graphs

Baker-Norine ('07)

Algebraic geometry	**Graph theory**
Riemann surface	Graph
Divisor	Chip configuration
Linear equivalence (i.e., equivalence mod principal divisors)	Sequence of chip-firing moves
Picard group	Critical group

and now simplicial complexes?

Algebraic geometry	**Simplicial complexes**
Variety	Simplicial complex
Algebraic cycle	Simplicial (co)chain
Rational equivalence (i.e., equiv. mod principal divisors)	Flow redistribution (i.e., equiv. mod Laplacians)
Chow group	Simplicial critical group
Chow ring	?????

In other words, how can we define a multiplication on elements (equivalence classes) of the simplicial critical group?