Matroids and statistical dependency

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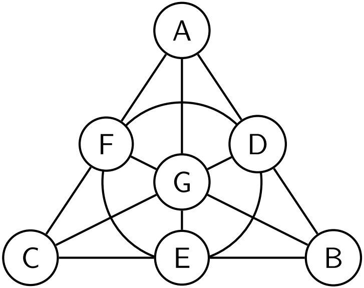
- Yes. For instance, $Z = 1 + XY + \epsilon$.
- We might expect to get any sort of simplicial complex (subsets of independent sets are independent).
- We can even get the Fano plane: A, B, C independent, $D = AB, E = BC, F = CA, G = DEF$.



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- In regression modeling, matroid structures could be used as a variable selection procedure to find the most parsimonious set of X's to predict a Y. The results of the matroid circuits would also inform which interactions ($x_1 x_2$ products) should be investigated for inclusion to the model.
- In big data settings, a matroid would identify maximally independent sets [bases] so that multiplicity can be corrected at the circuit level rather than the full data set.

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So when does this happen?

Closure axioms

A matroid on ground set E may be defined by closure axioms:

$$\mathrm{cl}\colon 2^E \to 2^E$$

- Closure axioms:
  - $A \subseteq \mathrm{cl}(A)$
  - If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
  - $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
- Exchange axiom: If $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$, then $y \in \mathrm{cl}(A \cup x)$

For us, $x \in \mathrm{cl}(A)$ means that knowing the values of all the variables in A implies knowing something about the value of x. (Sort of: x is a function of A, with statistical noise and fuzziness.)

Invertibility

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- $x \in \text{cl}(A \cup y) - \text{cl}(A)$ means that in using $A \cup y$ to determine x, we must use (can't ignore) y. ("model parsimony")
- $y \in \text{cl}(A \cup x)$ means we can "solve" for y in terms of x and A. (This is sort of invertibility.)

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Easiest way for a function (only way for continuous function) to be invertible is to be monotone in each variable. Fortunately, implied by a common statistical assumption:

Definition (MTP_2)

(Multivariate Totally Positive of order 2.)
$f(u)f(v) \leq f(u \wedge v)f(u \vee v)$, where f is probability distribution, u and v are vectors of variable values, and $\wedge$ and $\vee$ denote element-wise minimum and maximum.

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Example

When $A = x$ is a single element and $\text{cl}(x) = \{x, y\}$. We need to avoid $z \in \text{cl}\{x, y\}$ for $z \neq x, y$. In other words, z depends on y, and y depends on x should mean that z depends on x directly. This is a kind of transitivity.

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More generally, if Z is determined by $Y_1, \ldots, Y_p$, and each Y_i is determined by $X_1, \ldots, X_q$, then Z should be determined directly by $X_1, \ldots, X_q$. This is a kind of composition.

Remark

MTP_2 means the dependence will be strong enough to guarantee transitivity, and more generally composition.

Dependence axioms

How we actually show that we have a matroid. The dependent sets $\mathcal{D}$ in a matroid satisfy:

- $\emptyset \notin \mathcal{D}$
- If $D \in \mathcal{D}$ and $D' \supseteq D$, then $D' \in \mathcal{D}$
- If $I \notin \mathcal{D}$ but $I \cup x, I \cup y \in \mathcal{D}$, then $(I - z) \cup \{x, y\} \in \mathcal{D}$ for all $z \in I$.

We can prove that MTP_2 distributions satisfy this, using results of Fallat et al. (using that MTP_2 is an upward-stable singleton-transitive compositional semigraphoid).