Cuts and flows in cell complexes, I: Topology and vector space bases

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Special Session on Enumerative and Geometric Combinatorics
University of Kansas
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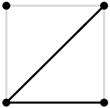
Cuts and bonds

Let G be a connected graph

Definition

A cut is a collection of edges in G whose removal disconnects the graph;

Example



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A cut is a collection of edges in G whose removal disconnects the graph; a bond is a minimal cut.

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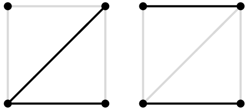

Remark

Using matroid language, bonds are cocircuits.

Orientation

Removal of a bond leaves two connected "shores" of the graph. This gives an alternative way to define a bond: describe the partition of the vertices of the graph.
We can orient all edges of the bond so they point from, say, north to south

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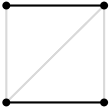

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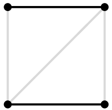

In fact, we can get this orientation from the partition: Take the coboundary (directed edges pointing out) of all vertices in, say, north shore. All edges completely within shore cancel out, leaving only those edges coming out of the north shore

Cut space

This suggests looking at cuts and bonds as the image of the coboundary. This is a vector space over field ($\mathbb{R}$ or $\mathbb{Q}$)

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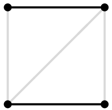

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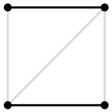

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What is a basis?

Fundamental bond

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Given a spanning tree T

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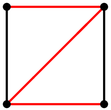

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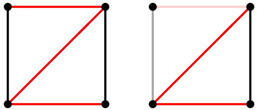

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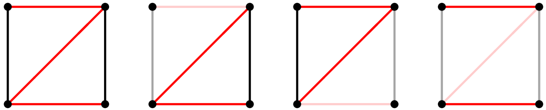

Theorem

For a fixed spanning tree, the collection of fundamental bonds forms a basis of cut space

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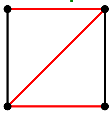

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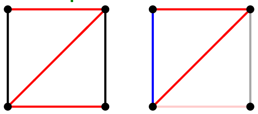

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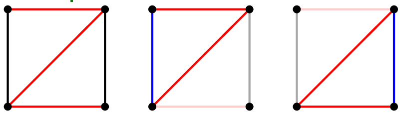

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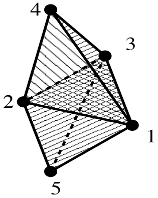
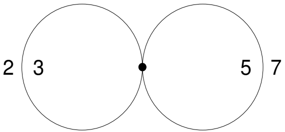
Remark

Any $\mathbb{Z}$ matrix can be the boundary matrix of a cell complex

Examples





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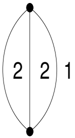

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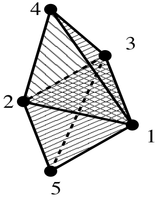
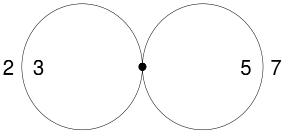
Cellular matroids

- Matroid whose elements are columns of boundary matrix





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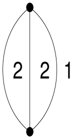

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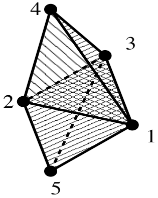

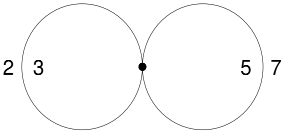

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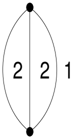

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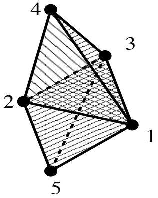
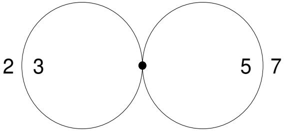
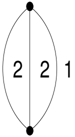
- Spanning trees are bases of these matroids (maximal independent sets)







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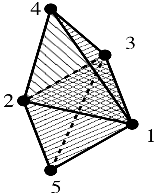

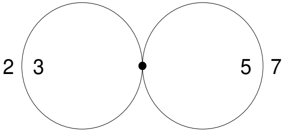

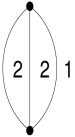

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i-dimensional cut space of cell complex X is

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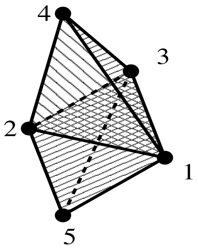
Bonds are the cocircuits of cellular matroid

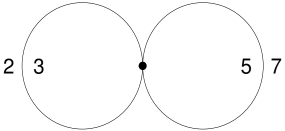
Topological interpretation of bonds

Remark

Bonds are minimal for increasing $(i-1)$-dimensional homology instead of decreasing i-dimensional homology

Examples





Characteristic vectors of bonds

Fix bond B

Proposition

$\mathrm{Cut}_B(X) := (\{0\} \cup (\mathrm{Cut}_i(X) \cap \{v : \mathrm{supp}(v) = B\}))$ is 1-dimensional

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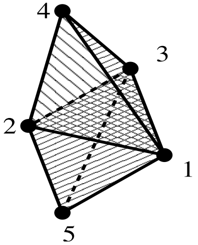

Topological interpretation of characteristic vector

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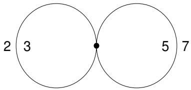

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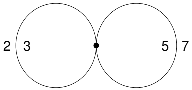

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Theorem (DKM)

Let A be a cellular spanning forest of X/B. Then $\text{Cut}_B(X)$ is spanned by

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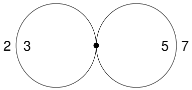

If $B = \{F_5, F_7\}$, then $\chi(B, F_2) = 2(5F_5 + F_7)$, but $\chi(B, F_3) = 3(5F_5 + F_7)$.

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The characteristic vector of B is $\chi(B, A)$

Orientation

One way to get at the sign in $\chi(B, A)$: Oriented matroid theory says that for every pair of elements F, F' of a cocircuit B, there is a unique circuit C such that $C \cap B = \{F, F'\}$, and we can deduce the relative signs on F, F' in χ by their relative signs in C.

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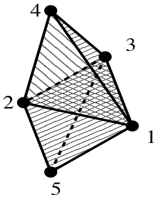

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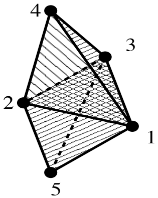

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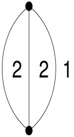

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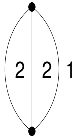

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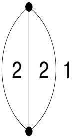

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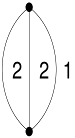


$$\begin{matrix} 0 & -2 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{matrix} \; ; \; \tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}; \; \chi(C) = (4, 2, 2)$$

Theorem (DKM)

$$\chi(C) = \sum_{F \in C} \pm |\mathbf{T}\tilde{H}(C \setminus F, \mathbb{Z})|$$

spans $\mathrm{Cut}_C(X)$*, where* $\mathbf{T}$ *stands for torsion part.*

Definition

The characteristic vector of C is $\chi(C)$

Orientation

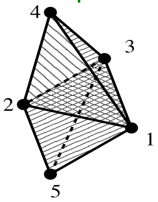
The orientation on the faces is given by the witness for the circuit being in the kernel of the boundary. But we can still use the observation on the signs for cocircuits if, for some reason, it's easier to get at the cocircuits than the circuits.

Fundamental circuit

Definition

Given a spanning forest Υ and an face $F \notin \Upsilon$, the fundamental circuit is the unique circuit in $\Upsilon \cup \{F\}$.

Example



$\Upsilon = \{124, 134, 123, 135, 235\}$

F	C
234	$\{123, 124, 134, 234\}$
235	$\{123, 125, 135, 235\}$

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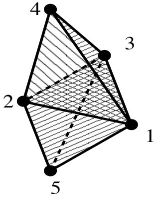

$\Upsilon = \{124, 134, 123, 135, 235\}$

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Theorem (DKM)

For a fixed spanning forest, the collection of fundamental circuits forms a basis of flow space