Eigenvalues of combinatorial Laplacians

MAA Southwestern Section

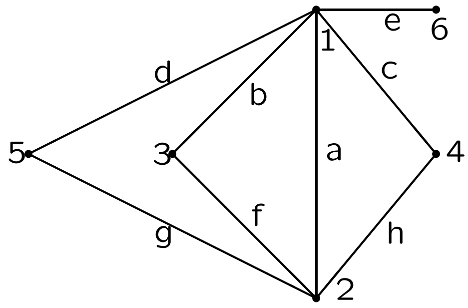
UTEP

Apr. '05

Eigenvalues of combinatorial Laplacians

Art Duval,
University of Texas at El Paso

GRAPHS

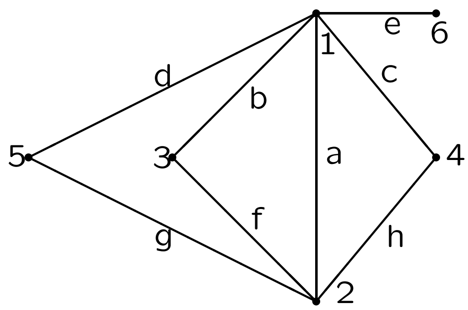


$$\partial_1 = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 3 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 5 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 6 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}$$

$$\partial_1 \partial_1^T = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 5 & -1 & -1 & -1 & -1 & -1 \\ 2 & -1 & 4 & -1 & -1 & -1 & 0 \\ 3 & -1 & -1 & 2 & 0 & 0 & 0 \\ 4 & -1 & -1 & 0 & 2 & 0 & 0 \\ 5 & -1 & -1 & 0 & 0 & 2 & 0 \\ 6 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$= D - A$$

EIGENVALUES



$$\partial_1 \partial_1^T = \begin{matrix} & 1 & \dots & 6 \\ 1 & & & \\ \vdots & & & \\ 6 & & & \end{matrix}$$

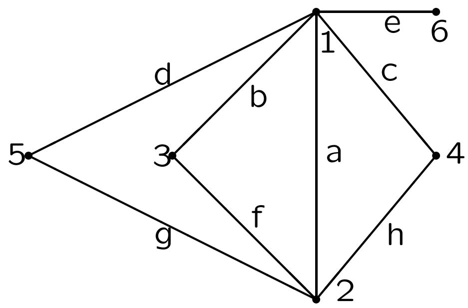
e'vals 652210

$$\partial_1^T \partial_1 = \begin{matrix} & a & \dots & h \\ a & & & \\ \vdots & & & \\ h & & & \end{matrix}$$

e'vals 65221000

$$\partial_1^T \partial_1 (\partial_1^T x) = \partial_1^T (\partial_1 \partial_1^T x) = \partial_1^T (\lambda x) = \lambda (\partial_1^T x)$$

SIMPLICIAL COMPLEXES



$$\partial_2^T = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ P & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ Q & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$$\partial_2^T \partial_2 = \begin{array}{c|cc} & P & Q \\ P & 3 & 1 \\ Q & 1 & 3 \end{array}$$ e'vals 42

$$\partial_2 \partial_2^T = \begin{array}{c|ccc} & a & \dots & h \\ a & & & \\ \vdots & & & \\ h & & & \end{array}$$ e'vals 42000000

LAPLACIAN

Defn: (1-dimensional) Laplacian

$$L_1 = \partial_1^T \partial_1 + \partial_2 \partial_2^T.$$

But now note $\partial_1 \partial_2 = 0$! (Because, for instance,

$$\begin{aligned}\partial_1(\partial_2(123)) &= \partial_1(12 - 13 + 23) \\ &= (1-2) - (1-3) + (2-3) = 0.)\end{aligned}$$

So

$$(\partial_1^T \partial_1)(\partial_2 \partial_2^T) = (\partial_2 \partial_2^T)(\partial_1^T \partial_1) = 0$$

and so the non-zero eigenvectors of $\partial_1^T \partial_1$ are 0-eigenvectors of $\partial_2 \partial_2^T$ and vice versa. (Because

$$(\partial_2 \partial_2^T)x = (\partial_2 \partial_2^T)(\partial_1^T \partial_1 x/\lambda) = 0.)$$

And then the multiset of eigenvalues of L_1 is just the multiset union of the eigenvalues of $\partial_1^T \partial_1$ and $\partial_2 \partial_2^T$.

So, in example, e'vals of L_1 are 65221 42 0

APPLICATIONS
(eigenvalues of graphs)

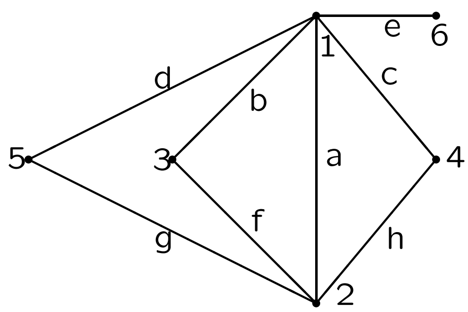
If graph is "grid graph," then $Lx = \lambda x$ is discrete version of (continuous) wave equation, using usual (continuous) Laplace operator (eigenvalues are notes of drum).

$\operatorname{rank} L = n - (\#\text{components})$.
So if graph is connected, then only one 0 eigenvalue (eigenvector is all 1's vector). In fact, second smallest eigenvalue is "algebraic connectivity," which is the right measure of connectivity sometimes.
Also, number of 0 eigenvalues counts number of cycles (actually, dimension of vector space of cycles); this generalizes to higher dimensions.

In chemical applications, eigenvalues correlate with physical properties (sometimes better than more widely used adjacency matrix).

SHIFTED COMPLEXES

If F is a face of a shifted complex, then any new face you get by replacing a bigger vertex in F by a smaller vertex must also be in the complex.

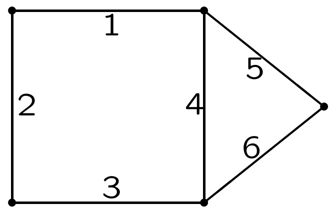


For example, because 25 is an edge, so is 13, 14, 15, 23, 24. And because 124 is a triangle, so is 123.

Note that this is not a total order; for instance 25 is incomparable to 16

MATROID COMPLEXES

Easiest way is to look at example, *graphic matroid*.



Faces of (graphic) matroid complex are all sets of edges containing no cycles. (Note: vertices of complex are edges of original graph!)

1,2,3,4,5,6
12,13,. . . ,56
123,124,. . . ,356, [not 456]
1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 2345, 2346, 2356

THEOREMS

(Kook-Reiner-Stanton '00) eigenvalues of matroid complexes are integers (nice formula)

(D-Reiner '02) eigenvalues of shifted complexes are integers (beautiful formula)

(D preprint '03) eigenvalues of shifted complexes and matroid complexes satisfy the *same* recursion (deletion; contraction; error term = deletion mod contraction).