The Critical group of a simplicial complex

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Sandpiles and chip-firing

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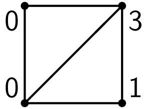

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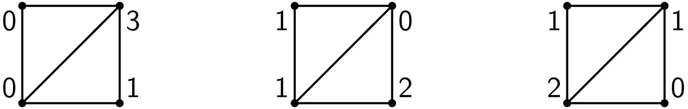

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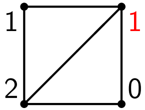

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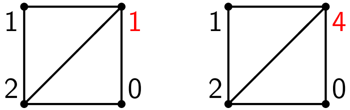

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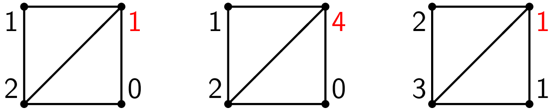

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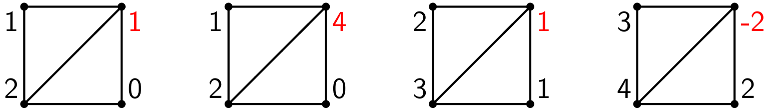

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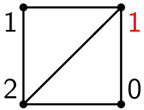

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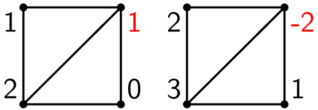

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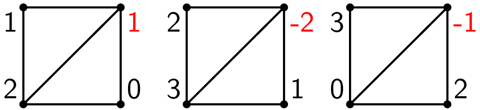
- A configuration is critical when it is stable and recurrent.



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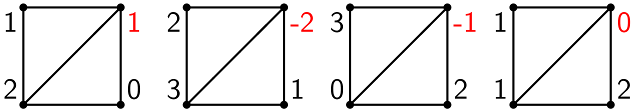

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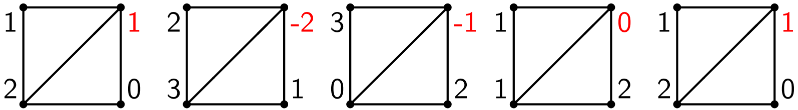
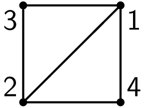


Fact: Every configuration topples to a unique critical configuration.

Laplacian

Let's make a matrix of how chips move when each vertex fires:



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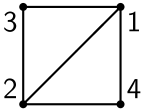

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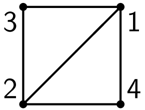

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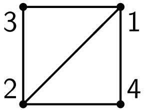

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So firing v is subtracting Lv (row/column v from L) from $(c_1, \ldots, c_n)$.

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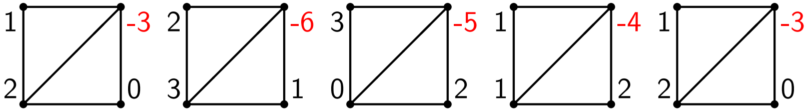

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- Recall every configuration is equivalent to a critical configuration.
- This equivalence means adding/subtracting integer multiples of Lv_i.
- In other words, instead of $\ker \partial$, we look at

$$K(G) := \ker \partial / \operatorname{im} L$$

the critical group. (It is a graph invariant.)

Reduced Laplacian and spanning trees

Theorem (Biggs '99)

$$K := (\ker \partial)/(\operatorname{im} L) \cong \mathbb{Z}^{n-1}/L_r,$$

where L_r denotes reduced Laplacian; remove row and column corresponding to source vertex.

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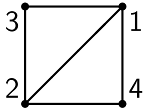
and $|\det L_r|$ counts spanning trees. □

Example



$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

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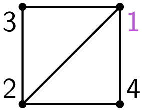

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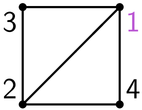

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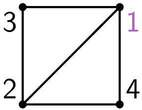

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$\det L_r = 8$, and there are 8 spanning trees of this graph

Generalize to simplicial complexes

Let Δ be a d-dimensional simplicial complex.

$$C_d(\Delta;\mathbb{Z}) \underset{\partial_d}{\overset{\partial_d^*}{\leftrightarrows}} C_{d-1}(\Delta;\mathbb{Z}) \xrightarrow{\partial_{d-1}} C_{d-2}(\Delta;\mathbb{Z}) \to \cdots$$

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where $L_{d-1} = \partial_d \partial_d^*$ is the $(d-1)$-dimensional up-down Laplacian.

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where $L_{d-1} = \partial_d \partial_d^*$ is the $(d-1)$-dimensional up-down Laplacian. Can we compute it with a reduced Laplacian? How do we reduce the Laplacian? And what about the trees?

Simplicial spanning trees of arbitrary simplicial complexes

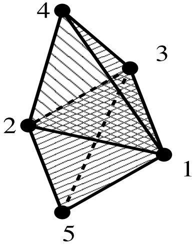
Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Let's figure out all its simplicial spanning trees.

Acyclic in Positive Codimension (APC)

- Denote by $\mathscr{T}(\Delta)$ the set of simplicial spanning trees of Δ.
- **Proposition** $\mathscr{T}(\Delta) \neq \emptyset$ iff Δ is **APC**, *i.e.* (equivalently)
  - homology type of wedge of spheres;
  - $\tilde{H}_j(\Delta; \mathbb{Z})$ is finite for all $j < \dim \Delta$.
- Many interesting complexes are APC.

Simplicial Matrix-Tree Theorem

- Δ a d-dimensional APC complex
- $\Gamma \in \mathcal{T}(\Delta_{(d-1)})$
- ∂_{Γ} = restriction of ∂_d to faces not in Γ
- reduced (up-down) $(d-1)$-dimensional Laplacian $L_\Gamma = \partial_\Gamma \partial^*_\Gamma$

Theorem [DKM '09]

$$h_d = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma .$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

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$\Gamma = 12, 13, 14, 15$ spanning tree of 1-skeleton

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$L_\Gamma =$

	23	24	25	34	35
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$\det L_\Gamma = 15.$

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Spanning trees

Theorem (DKM)

$$K(\Delta) := (\ker \partial_{d-1})/(\operatorname{im} L_{d-1}) \cong \mathbb{Z}^r / L_\Gamma$$

where Γ *is a torsion-free* $(d-1)$*-dimensional spanning tree and* $r = \dim L_\Gamma$.

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Corollary

$|K(\Delta)|$ *is the torsion-weighted number of* d*-dimensional spanning trees of* Δ.

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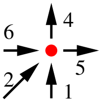

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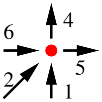
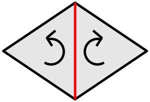
- conservative flow
  - $d = 2$: chips do not accumulate or deplete at any vertex;
  - $d = 3$: face circulation at each edge adds to zero.

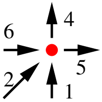


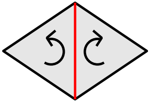


What does it look like?

$$K(\Delta) := \ker \partial_{d-1} / \operatorname{im} L_{d-1} \subseteq \mathbb{Z}^m$$

- Put integers on $(d-1)$-faces of Δ. Orient faces arbitrarily. $d = 2$: flow; $d = 3$: circulation; etc.
- conservative flow
  - $d = 2$: chips do not accumulate or deplete at any vertex;
  - $d = 3$: face circulation at each edge adds to zero.
- By theorem, just specify values off the spanning tree.

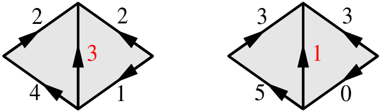




Firing faces

$$K(\Delta) := \ker \partial_{d-1} / \operatorname{im} L_{d-1} \subseteq \mathbb{Z}^m$$

Toppling/firing moves the flow to “neighboring” $(d-1)$-faces, across d-faces.



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- But this misses the sense of "critical".
- Main obstacle is idea of what is "positive".