Spanning trees and the critical group of simplicial complexes

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Spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

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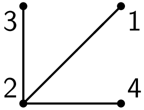
$T \subseteq E(K_n)$ is a **spanning tree** of K_n when:

0. spanning: T contains all vertices;
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

Example: K_4

- 4 trees like: $T =$ 

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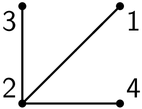
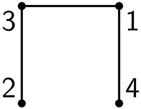
- 4 trees like: $T =$ 

- 12 trees like: $T =$ 

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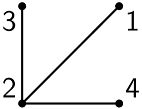
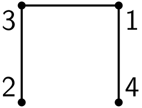
- 4 trees like: $T =$ 

- 12 trees like: $T =$ 

Total is $16 = 4^2$.

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Definition The Laplacian matrix of graph G, denoted by $L(G)$.

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$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

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Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

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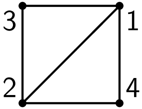
"Reduced": remove rows/columns corresponding to any one vertex

Example

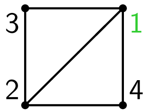


$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Example



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Matrix-Tree Theorems

Version I Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then G has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

spanning trees.

Version II G has $|\det L_r(G)|$ spanning trees

Proof [Version II]

$$\det L_r(G) = \det \partial_r(G) \partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$
$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n-1 \times n-1)$$

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Version II:

$$\begin{aligned} \det L_r &= \prod \text{eigenvalues} \\ &= (n-0)^{(n-1)-1}(n-(n-1)) \\ &= n^{n-2} \end{aligned}$$

Complete skeleta of simplicial complexes

Simplicial complex $\Delta \subseteq 2^V$;

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Complete skeleton The d-dimensional complete complex on n vertices, *i.e.*,

$$K_n^d = \{F \subseteq V \colon |F| \leq d + 1\}$$

(so $K_n = K_n^1$).

Simplicial spanning trees of K_n^d [Kalai, '83]

$\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^d)} \qquad = n^{\binom{n-2}{d}}$$

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$$\sum_{\Upsilon \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}$$

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Proof uses determinant of reduced Laplacian of K_n^d. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(d-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial \colon \Delta_d \to \Delta_{d-1}$ boundary

$\partial^T \colon \Delta_{d-1} \to \Delta_d$ coboundary

Example $n = 4, d = 2$

$\partial^T =$

	12	13	14	23	24	34
123	-1	1	0	-1	0	0
124	-1	0	1	0	-1	0
134	0	-1	1	0	0	-1
234	0	0	0	-1	1	-1

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Simplicial spanning trees of arbitrary simplicial complexes

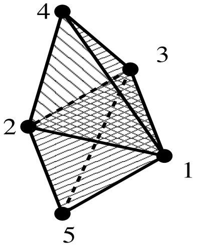
Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Let's figure out all its simplicial spanning trees.

Acyclic in Positive Codimension (APC)

- Denote by $SST(\Delta)$ the set of simplicial spanning trees of Δ.
- **Proposition** $SST(\Delta) \neq \emptyset$ iff Δ is **APC**, *i.e.* (equivalently)
  - homology type of wedge of spheres;
  - $\tilde{H}_j(\Delta; \mathbb{Z})$ is finite for all $j < \dim \Delta$.
- Many interesting complexes are APC.

Simplicial Matrix-Tree Theorem — Version I

- Δ a d-dimensional APC simplicial complex
- $(d-1)$-dimensional **(up-down) Laplacian** $L_{d-1} = \partial_{d-1}\partial_{d-1}^T$
- s_d = product of nonzero eigenvalues of L_{d-1}.

Theorem [DKM '09]

$$h_d := \sum_{\Upsilon \in SST(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Delta)|^2$$

Simplicial Matrix-Tree Theorem — Version II

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial^T{}_\Gamma$

Theorem [DKM '09]

$$h_d \;=\; \sum_{\Upsilon \in SST(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Bipyramid again

$\Gamma = 12, 13, 14, 15$ spanning tree of 1-skeleton

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$L_\Gamma =$

	23	24	25	34	35
23	3	-1	-1	1	1
24	-1	2	0	-1	0
25	-1	0	2	0	-1
34	1	-1	0	2	0
35	1	0	-1	0	2

Bipyramid again

$\Gamma = 12, 13, 14, 15$ spanning tree of 1-skeleton

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$\det L_\Gamma = 15.$

Sandpiles and chip-firing

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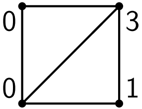
Abstraction Graph G with vertices $v_1, \ldots, v_n$. Degree of v_i is d_i. Place $c_i \in \mathbb{Z}$ chips (grains of sand) on v_i.



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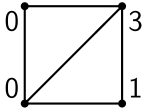
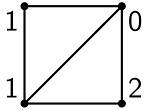
Toppling If $c_i \geq d_i$, then v_i may fire by sending one chip to each of its neighbors.





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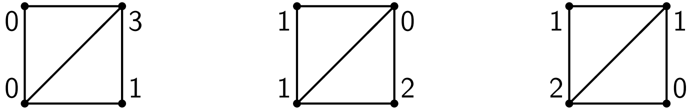
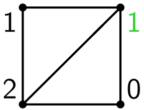


Source vertex

- To keep things going, pick one vertex v_r to be a source vertex. We can always add chips to v_r.



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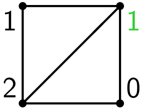

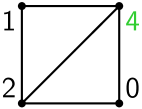

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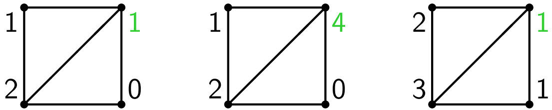

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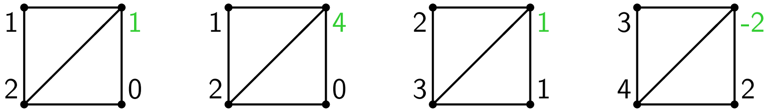

Critical configurations

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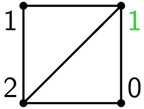

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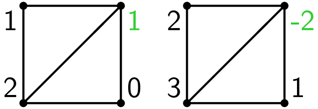
- A configuration is recurrent when a series of topplings leads back to that configuration, without letting any vertex (except the source vertex) go negative.



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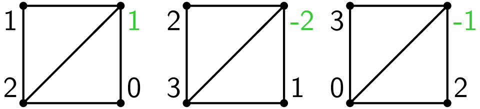
- A configuration is critical when it is stable and recurrent.



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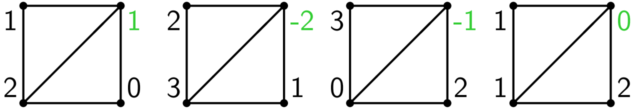

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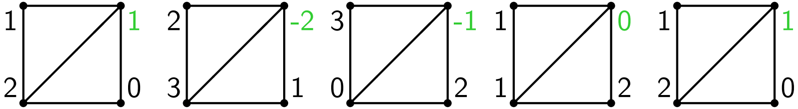
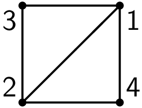


Fact: Every configuration topples to a unique critical configuration.

Laplacian

Let's make a matrix of how chips move when each vertex fires:



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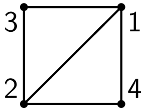

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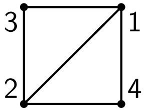

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where

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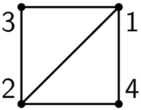

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So firing v is subtracting Lv (row/column v from L) from $(c_1, \ldots, c_n)$.

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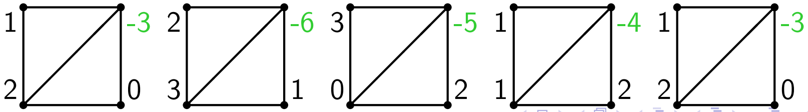

Critical group

- Consider two configurations (in $\ker \partial_0$) to be equivalent when you can get from one to the other by chip-firing.

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- In other words, instead of $\ker \partial_0$, we look at

$$K(G) := (\ker \partial_0)/(\operatorname{im} L) = (\ker \partial_0)/(\operatorname{im} \partial_1 \partial_1^T)$$

the critical group. (It is a graph invariant.)

Reduced Laplacian and spanning trees

Theorem (Biggs '99)

$$K := (\ker \partial_0)/(\operatorname{im} L) \cong \mathbb{Z}^{n-1}/(\operatorname{im} L_r),$$

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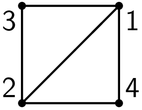
and $|\det L_r|$ counts spanning trees. □

Example



$\partial_1 =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
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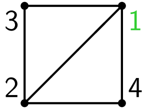

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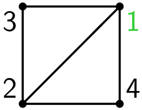

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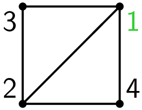

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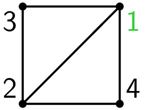

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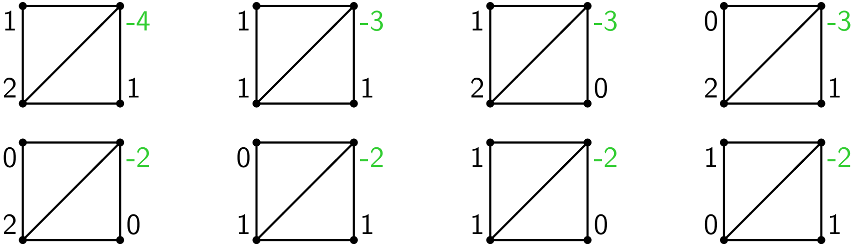
det $L_r = 8$, and there are 8 spanning trees of this graph and 8 critical configurations:



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So let's generalize critical groups to simplicial complexes, and see if they can be computed by reduced Laplacians.

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$$K(\Delta) := (\ker \partial_{d-1})/(\operatorname{im} L_{d-1}),$$

where $L_{d-1} = \partial_d \partial_d^T$ is the $(d-1)$-dimensional up-down Laplacian.

Spanning trees

Theorem (DKM, pp '11)

$$K(\Delta) := (\ker \partial_{d-1})/(\operatorname{im} L_{d-1}) \cong \mathbb{Z}^t / L_\Gamma$$

where Γ is a torsion-free $(d-1)$-dimensional spanning tree, L_Γ is the reduced Laplacian (restriction to faces not in Γ), and $t = \dim L_\Gamma$.

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Corollary

$|K(\Delta)|$ is the torsion-weighted number of d-dimensional spanning trees of Δ.

Proof.

$|K(\Delta)| = |(\mathbb{Z}^t)/L_\Gamma| = |\det L_\Gamma|$, which counts (torsion-weighted) spanning trees. □

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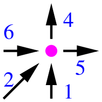

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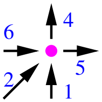

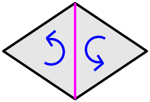

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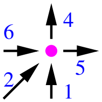
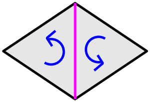
- By theorem, just specify values off the spanning tree.

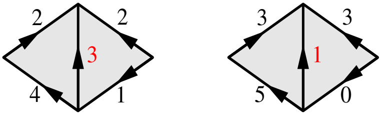




Firing faces

$$K(\Delta) := (\ker \partial_{d-1})/(\text{im}\, L_{d-1}) \subseteq \mathbb{Z}^m$$

Toppling/firing moves the flow to "neighboring" $(d-1)$-faces, across d-faces.



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- Main obstacle is idea of what is "positive".

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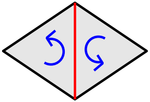

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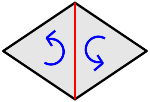

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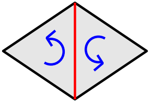

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- So $K(\Delta)$ has a single generator, so it is cyclic.
- $|K(\Delta)|$ is the number of spanning trees, and there is one tree for every facet (remove that facet for the tree).

□

Riemann-Roch Theorem for Graphs

Baker-Norine ('07)

Algebraic geometry	**Graph theory**
Riemann surface	Graph
Divisor	Chip configuration
Linear equivalence (i.e., equivalence mod principal divisors)	Sequence of chip-firing moves
Picard group	Critical group

and now simplicial complexes?

Algebraic geometry	**Simplicial complexes**
Variety	Simplicial complex
Algebraic cycle	Simplicial (co)chain
Rational equivalence (i.e., equiv. mod principal divisors)	Flow redistribution (i.e., equiv. mod Laplacians)
Chow group	Simplicial critical group
Chow ring	?????

In other words, how can we define a multiplication on elements (equivalence classes) of the simplicial critical group?

Final thought

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