The G-Shi arrangement, and its relation to G-parking functions

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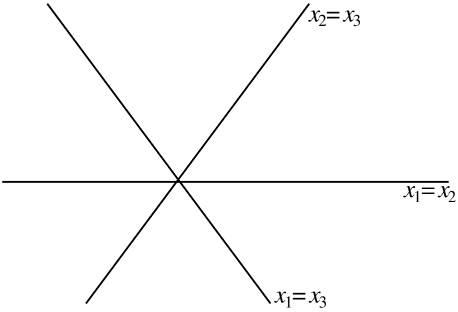
Arrangements

Braid $B_n := \{x_i = x_j \qquad : 1 \leq i < j \leq n\}$ $\quad n!$ regions



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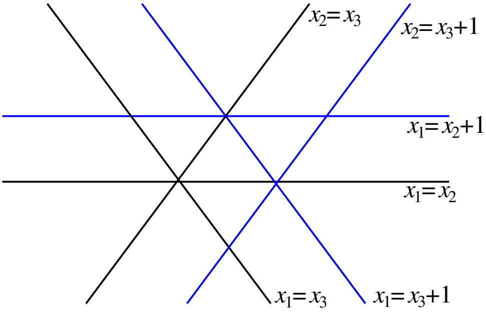
Shi $\mathcal{S}_n := \{x_i = x_j, x_i = x_j + 1 : 1 \leq i < j \leq n\}$ $(n+1)^{n-1}$ regions



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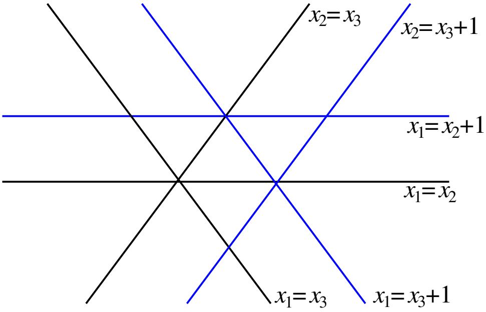


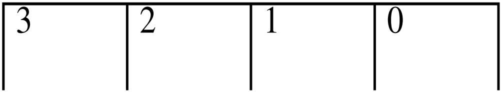
$(n+1)^{n-1}$ is also the number of spanning trees of K_n (Cayley)

Parking functions

Definition

- parking spots $0, \ldots, n-1$

Example



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- parking spots $0, \ldots, n-1$
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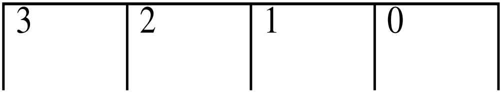

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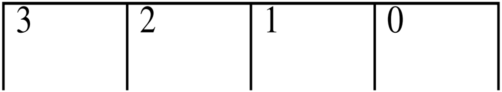
1120



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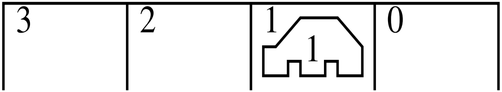

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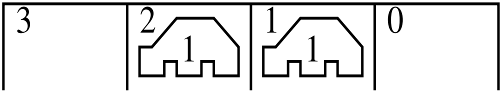

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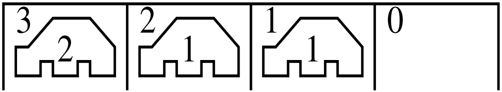

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- car i has favorite parking spot $f(i)$
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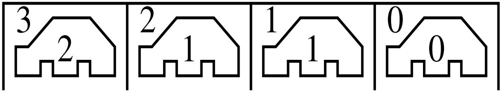
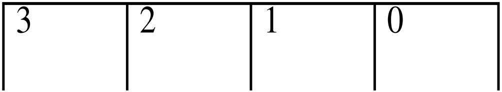
If such a function f allows all the cars to park, it is a **parking function**. [Note that indexing is sometimes different.]

Example

1120



Which functions are parking functions?



Example

Easy: All 0's;

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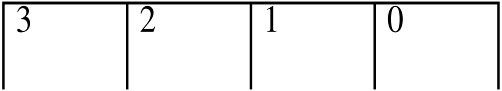

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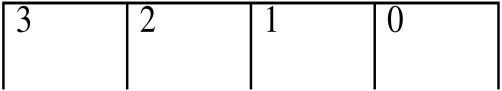

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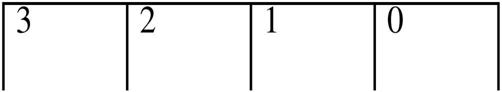

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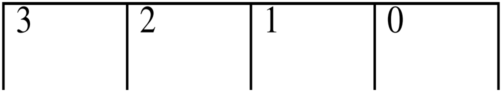

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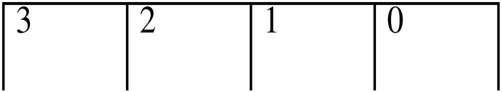

Example

Easy: All 0's; any permutation of $0, \dots, n-1$.

Example

These are not parking functions: 3003, 2230, 1121

Necessary: Fewer than i cars whose value is greater than $n - i$

Which functions are parking functions?

3	2	1	0

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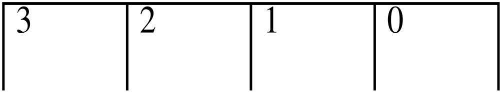

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Equivalently, when values $f(i)$ rearranged in increasing order, $f(i) < i$. (f is "componentwise" less than permutation of $0, \ldots, n-1$.)
This is sufficient, too (making values less only makes it easier to park).

How many are there?

n=2: 00, 01, 10

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n=3: 000, 001, 010, 100, 011, 101, 110, 002, 020, 200, 012, 021, 102, 120, 201, 210

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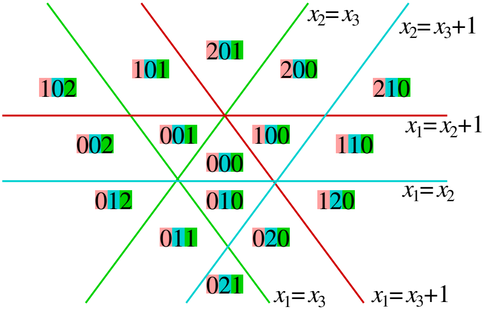
Theorem (Pyke, '59; Konheim and Weis, '66)

There are $(n+1)^{n-1}$ parking functions.

Pak-Stanley labelling

Pak and Stanley found a labelling of the regions of the Shi arrangement so that each region gets a different label,



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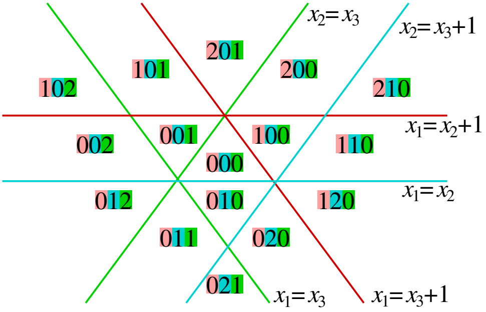

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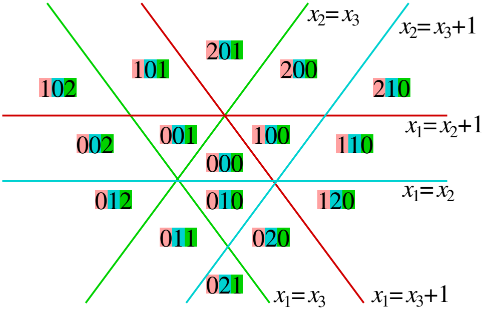

Athanasiadis and Linusson have alternate (easier) bijection between parking functions and Shi regions.

Restating parking function definition

Recall the original necessary and sufficient condition:

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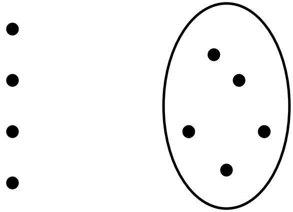
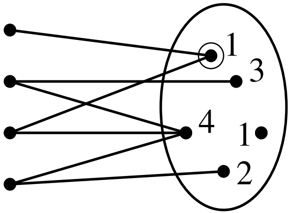


G-parking functions

Definition (Postnikov-Shapiro '04)

Given a graph $G = (V, E)$, , a function $f : V \quad \to \mathbb{Z}^{\geq 0}$ is a **parking function** if, in any set $U \subseteq V$ of vertices, there is at least one vertex v such that $f(v)$ is at most the $\bar{U}$-degree of v, the number of neighbors of v outside of U.



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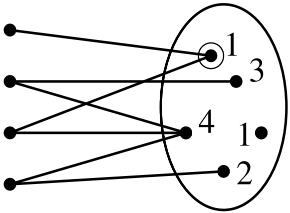
Definition (Postnikov-Shapiro '04)

Given a graph $G = (V, E)$, with root q, a function $f : V \backslash q \to \mathbb{Z}^{\geq 0}$ is a **parking function** if, in any set $U \subseteq V \setminus q$ of vertices, there is at least one vertex v such that $f(v)$ is at most the $\bar{U}$-degree of v, the number of neighbors of v outside of U.

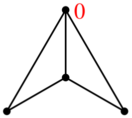


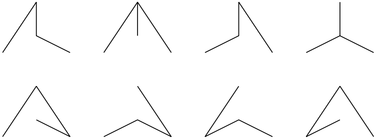
Note that if $G = K_{n+1}$ we get classical parking functions on n cars.

Spanning Trees

Theorem (Postnikov-Shapiro)

$\#\{G\text{-}parking\ functions\} = \#\{spanning\ trees\ of\ G * 0\}$.





0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
1	1	0
0	2	0

Graphical arrangement

Start with braid arrangement, but include only hyperplanes corresponding to edges in graph:

$$\{x_i = x_j \colon i < j; \{i, j\} \in E\}$$

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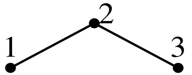
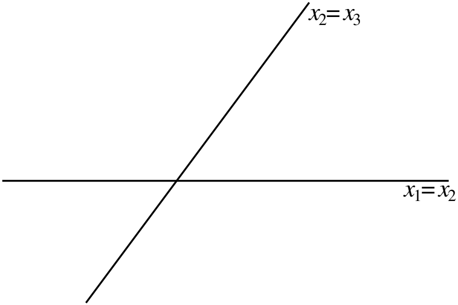
Example

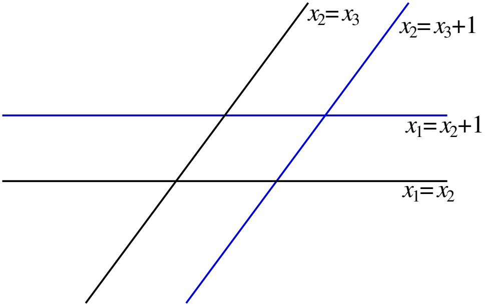




G-Shi arrangement

If we combine the ideas of the graphical arrangement and the Shi arrangement, we get

$$\{x_i = x_j, x_i = x_j + 1 \colon i < j; \{i, j\} \in E\}$$



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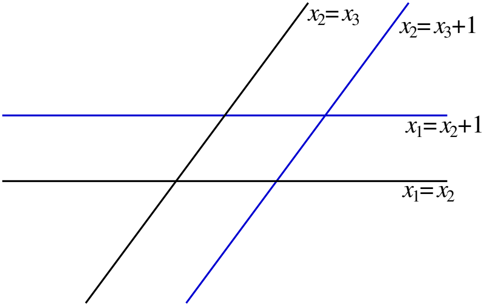
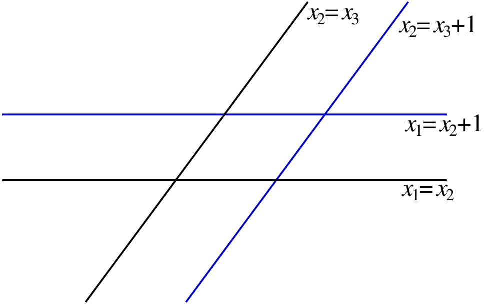


But this has 9 regions, and there are only 8 spanning trees and 8 parking functions.

Conjecture

0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
1	1	0
0	2	0



Conjecture

0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
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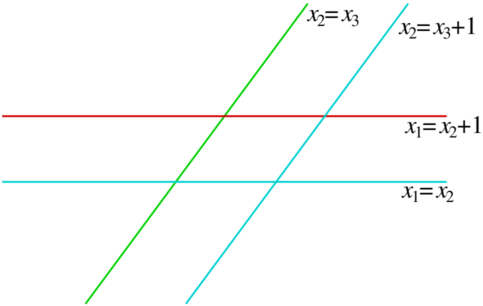

Conjecture

0 0 0
0 0 1
0 1 0
1 0 0
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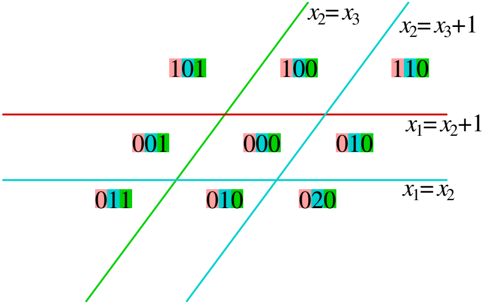
1 1 0
0 2 0



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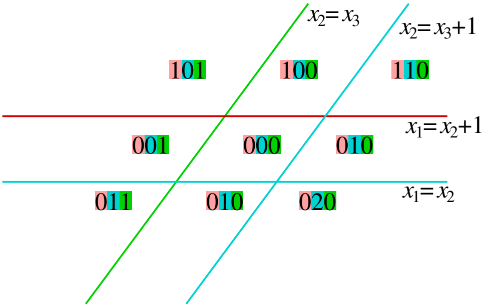
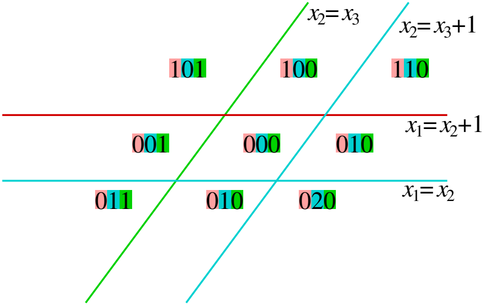


Conjecture

There is a bijection between the $(G * 0)$-parking functions and the set of different labels of the G-Shi arrangement.

Maximal labels in G-Shi

What are the maximal labels (maximum total weight) of the G-Shi arrangement?



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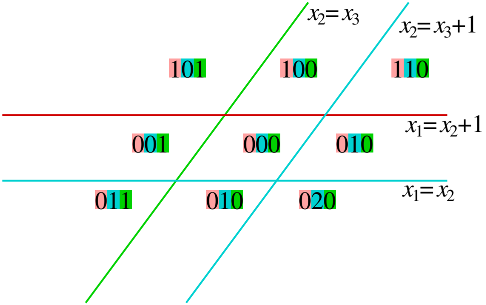
The regions can't be in any of the "middle slices"



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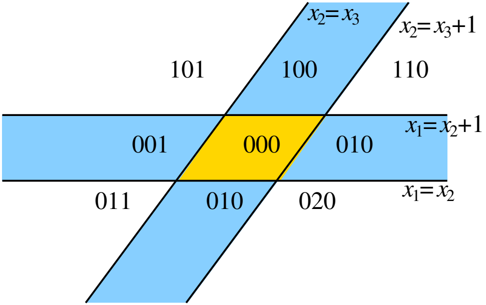
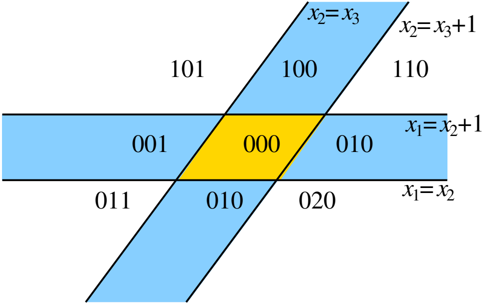


Graphical arrangement



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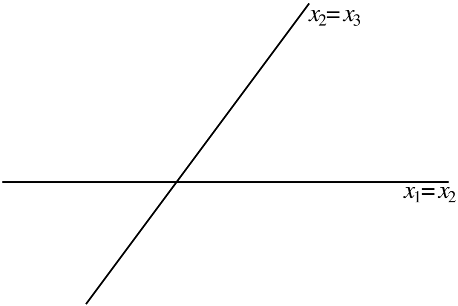
Graphical arrangement



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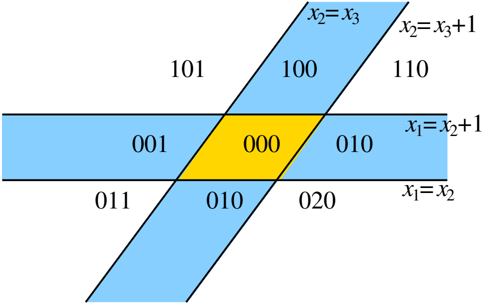
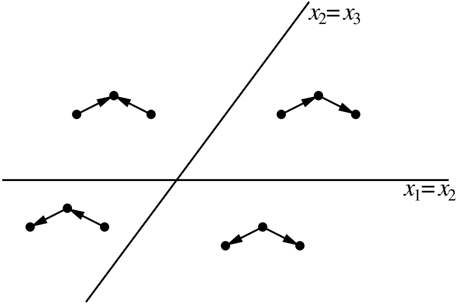


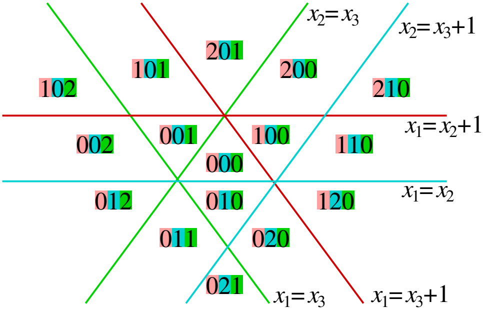
For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other: Graphical arrangement!
Weight goes up by one for every hyperplane crossed, so total weight is number of edges of G.

Acyclic orientations



Regions of graphical arrangement correspond to acyclic orientations on graph (just like regions of braid arrangement correspond to permutations, which correspond to acyclic orientations of the complete graph).
So there is a natural bijection between maximal labels of the G-Shi arrangement and acyclic orientations of G.

Example: K_n again



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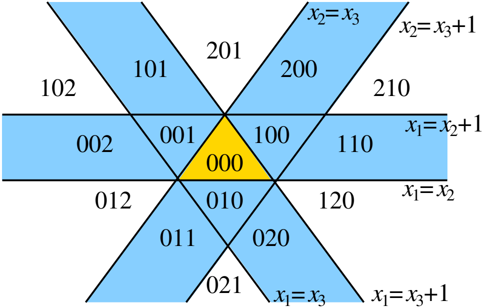

Maximal G-parking functions

Theorem (Benson, Chakrabarty, Tetali, '10)

Maximal $(G * 0)$-parking functions also have weight equal to the number of edges of G, and correspond to acyclic orientations of G.

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Observation (Easy)

If f is a G-parking function, and $g(v) \leq f(v)$ for all v, then g is also a G-parking function

Proof.

Reducing the values of the parking function can only make it easier to satisfy the condition. □

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Consequence: If we could only show that labels also satisfy the easy observation, we'd be done.

Half the bijection

We can use this to easily show that every label g has a corresponding parking function:
There exists some maximal label f such that $g(v) \leq f(v)$ for all v ($g = f$ is possible). Since f is maximal, it corresponds to an acyclic orientation O. By BCT, we know O corresponds to a maximal parking function, so f is a maximal parking function. By the easy observation, g is also a parking function.

What about the other half?

We still need to show either [equivalently]:

- Every parking function is a label
- Labels satisfy the easy observation