A Relative Laplacian spectral recursion

CombinaTexas

Texas State University, San Marcos
February, '05

A Relative Laplacian spectral recursion

Art Duval,
University of Texas at El Paso

OVERVIEW

The **eigenvalues** of the **combinatorial Laplacian** of the independence complexes of **matroids** and of **shifted** complexes are **integral**, with combinatorial formulas. (KRS '00; DR '02)

For "nice" **relative pairs** of matroids and shifted complexes, there are nice formulas, too. (D '03)

These eigenvalues satisfy the **same** nice **recursion** for both matroids and shifted complexes. (D '03)

Theorem: This recursion works for the "nice" relative pairs as well, using the "right" definition of each term of the recursion in the relative case. (new)

SHIFTED FAMILIES AND COMPLEXES

Shifted family $\mathcal{K}$: non-empty family of k-subsets of ground set $E = \{1, \ldots, n\}$ satisfying

$\forall F \in \mathcal{K},\ \forall v \in F$, $\forall v' < v,$ if $v' \notin F,$ then

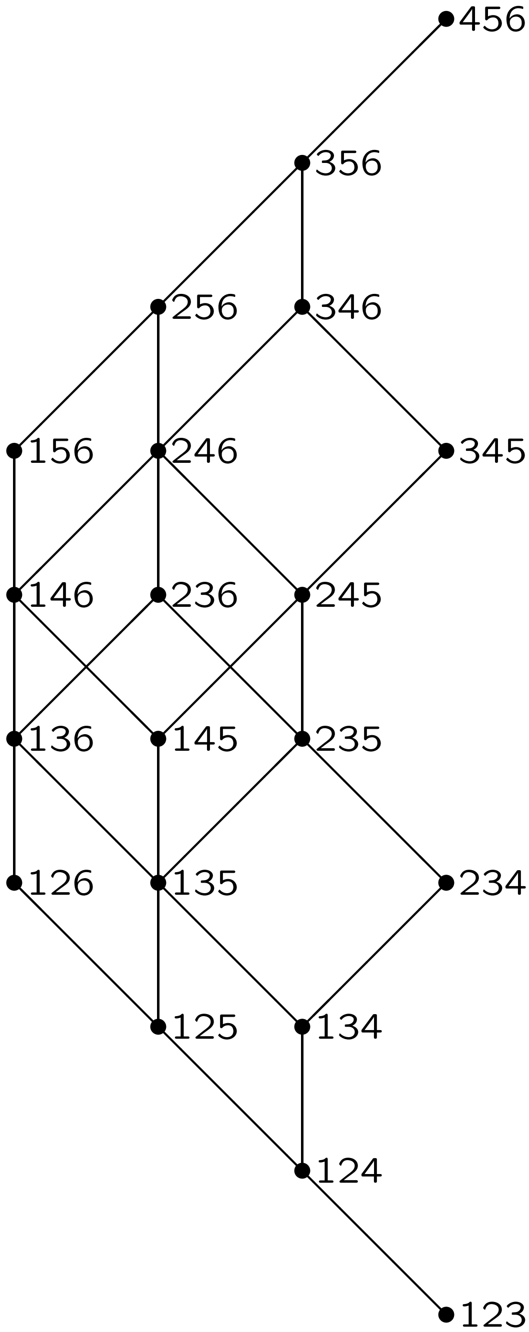
$$(F - v) \cup v' \in \mathcal{K}.$$

Example: 123, 124, 125, 126, 134, 135, 136, 145, 234, 235, 236.

A simplicial complex is shifted if its family of i-dimensional faces is shifted, for all i.

The simplicial complex formed by taking all subsets of every set $F \in \mathcal{K}$ is a pure shifted simplicial complex.

ORDER IDEAL



456
356
256
346
156
246
345
146
236
245
136
145
235
126
135
234
125
134
124
123

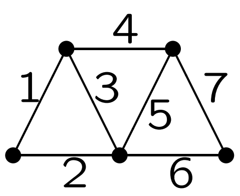
MATROIDS

Bases $\mathcal{B}$: non-empty family of k-subsets of ground set $E = \{1, \ldots, n\}$ satisfying

$\forall B \in \mathcal{B},\ \forall b \in B,\ \forall B' \in \mathcal{B},\ \exists b' \in B'$ such that

$$(B - b) \cup b' \in \mathcal{B}.$$

Example:



$\mathcal{B} =$

$(3 \in B)$		$(3 \notin B)$		
1346	2346	1246	1456	1467
1347	2347	1247	1457	2467
1356	2356	1256	2456	
1357	2357	1257	2457	
1367	2367	1267		

The simplicial complex formed by taking all subsets of every base $B \in \mathcal{B}$ is the set of independent sets IN(M) of matroid M.

RELATIVE PAIRS OF COMPLEXES

If $\Delta' \subseteq \Delta$ are a simplicial complexes on the same set of vertices, then $\Phi = (\Delta, \Delta') := \Delta - \Delta'$ is a relative pair of complexes.

When $\Delta' = \emptyset$, then $\Phi = (\Delta, \emptyset) = \Delta$.

Φ is an interval in the Boolean algebra.

LAPLACIANS

$C_i = C\Phi_i$, the i-dimensional $\mathbb{R}$-chains of Φ
($\mathbb{R}$-linear combinations of i-dim'l faces of Φ)

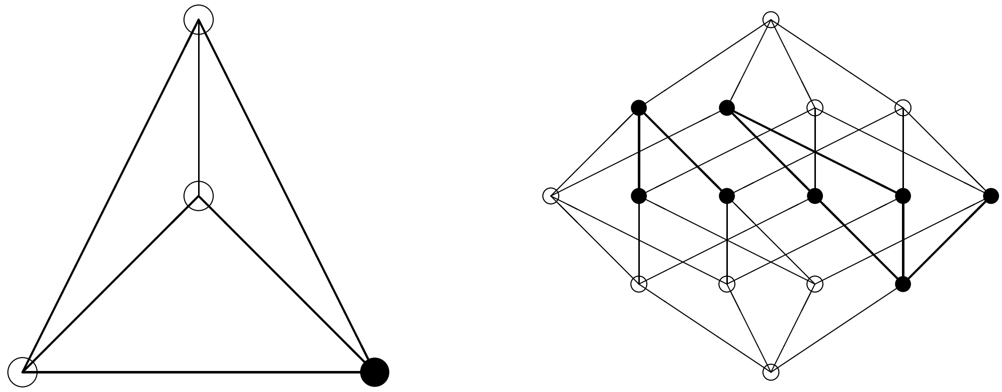
$\partial = \partial_i \colon C_i \to C_{i-1}$ usual signed boundary
$\delta_{i-1} = \partial_i^* \colon C_{i-1} \to C_i$ coboundary.

$$C_{i+1} \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_i \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_{i-1}$$

Defn: i-dimensional **Laplacian** of Φ:

$$L_i(\Phi) = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i \colon C_i \to C_i$$

Example:

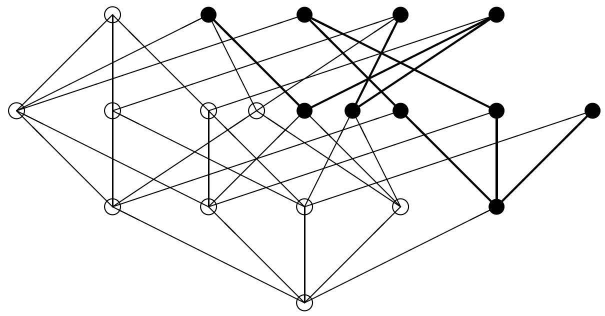


EIGENVALUES OF LAPLACIANS

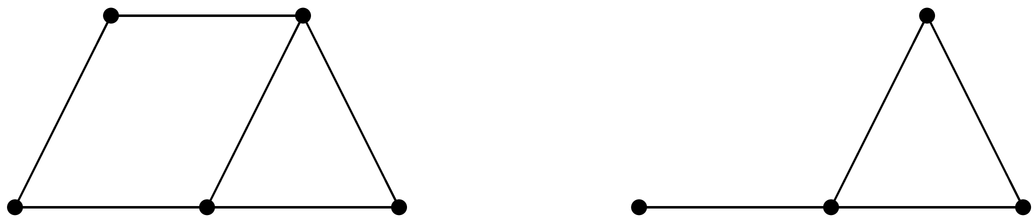
Let $\mathbf{s}(L_i)$ denote multiset of eigenvalues of L_i. Define a natural generating function:

$$S_\Phi(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Phi))} q^\lambda$$

E'vals are integers(!) w/nice formulas(!) for: pairs of shifted complexes with the *same vertex ordering* (D-Reiner '02; D '03);



pairs of matroids related by a *strong map* (Kook-Reiner-Stanton '00; D '03).



SPECTRAL RECURSION FOR MATROIDS...

Tutte polyn. deletion-contraction recursion:

$$T_M = T_{M-e} + T_{M/e}$$

$$\mathcal{B}(M-e) = \{B \in \mathcal{B} \colon e \notin B\} \qquad (r = r(M))$$
$$\mathcal{B}(M/e) = \{B - e \colon B \in \mathcal{B},\ e \in B\} \quad (r = r(M) - 1)$$

Thm (Kook): $S_M = qS_{M-e} + qtS_{M/e}$
$+(1-q)$(error term).

Conj(Kook-Reiner): error term $= S_{(M-e,M/e)}$,
where $(M-e, M/e) = (\mathrm{IN}(M-e), \mathrm{IN}(M/e))$.

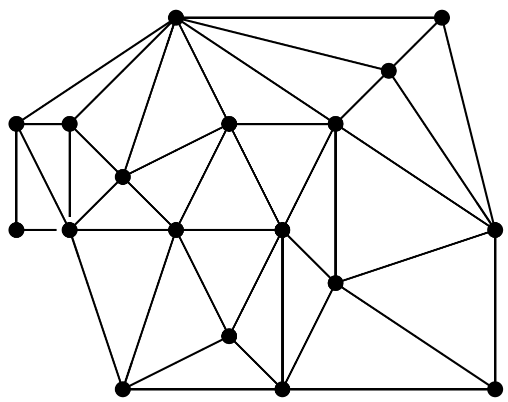
Thm (D '03): This is true, *i.e.*,

$$S_M = qS_{M-e} + qtS_{M/e} + (1-q)S_{(M-e,M/e)}.$$

... AND FOR SHIFTED COMPLEXES

Generalize deletion and contraction to arbitrary simplicial complex Δ.

$$\Delta - e = \{F \in \Delta \colon e \notin F\}$$
$$\Delta / e = \{F - e \colon F \in \Delta,\ e \in F\} \quad = \mathrm{lk}_\Delta\, e$$



$$S_\Delta(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Delta))} q^\lambda$$

Thm (D '03): Spectral recursion holds for shifted complexes Δ:

$$S_\Delta = qS_{\Delta - e} + qtS_{\Delta/e} + (1-q)S_{(\Delta - e, \Delta/e)}.$$

RELATIVE RECURSION

Say $\Phi = (\Delta, \Delta')$. Define

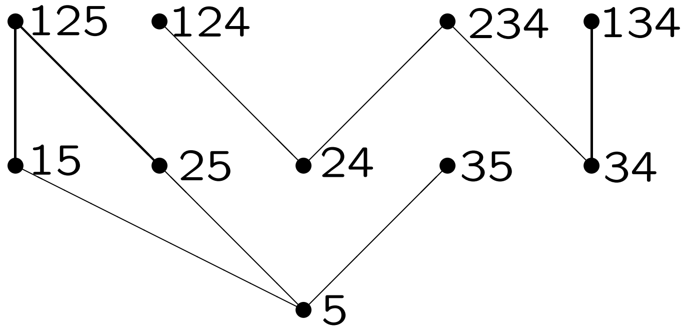
$$\begin{aligned} \Phi - e &= \{F \in \Phi \colon e \notin F\} &&= (\Delta - e, \Delta' - e) \\ \Phi/e &= \{F - e \colon F \in \Phi,\ e \in F\} &&= (\Delta/e, \Delta'/e) \end{aligned}$$

$$\begin{aligned} \Phi \parallel e &= \Phi - \{(F, F \,\dot\cup\, e) \colon (F, F \,\dot\cup\, e) \subseteq \Phi\} \\ &\approx (\Delta - e, (\Delta' - e) \cup \Delta/e) \\ &\quad \dot\cup\, (((\Delta' - e) \cap \Delta/e), \Delta'/e) \end{aligned}$$

Thm: If Φ is shifted pair (same vertex ordering) or matroid pair $(M - f, M/f)$, then

$$S_\Phi = q S_{\Phi - e} + qt S_{\Phi/e} + (1 - q) S_{\Phi \parallel e}$$

Example:



MORE ABOUT $\Phi \parallel e$

Original description of $(\Delta - e, \Delta / e)$ was $(\Delta, \mathrm{st}_\Delta\, e)$ (they are the same). In some sense, $\Phi \parallel e$ is $(\Phi, \mathrm{st}_\Phi\, e)$.

When plugging in $q = 0$, S is generating function for homology Betti numbers. $(\Delta, \mathrm{st}_\Delta\, e)$ has same homology as Δ, since $\mathrm{st}_\Delta\, e$ is contractible, so recursion for Δ is trivially true for all Δ. Same is true for Φ; note $\mathrm{st}_\Phi\, e = (\mathrm{st}_\Delta\, e, \mathrm{st}_{\Delta'}\, e)$.