A Relative Laplacian spectral recursion

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A Relative Laplacian spectral recursion

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OVERVIEW

The **eigenvalues** of the **combinatorial Laplacian** of the independence complexes of **matroids** and of **shifted** complexes are **integral**, with combinatorial formulas. (KRS '00; DR '02)

For "nice" **relative pairs** of matroids and shifted complexes, there are nice formulas, too. (D '03)

These eigenvalues satisfy the **same** nice **recursion** for both matroids and shifted complexes. (D '03)

Conjecture: This recursion works for "nice" relative pairs as well, using the "right" definition of each term of the recursion in the relative case. (new)

SHIFTED FAMILIES AND COMPLEXES

Shifted family $\mathcal{K}$: non-empty family of k-subsets of ground set $E = \{1, \ldots, n\}$ satisfying

$\forall F \in \mathcal{K},\ \forall v \in F$, $\forall v' < v,$ if $v' \notin F,$ then

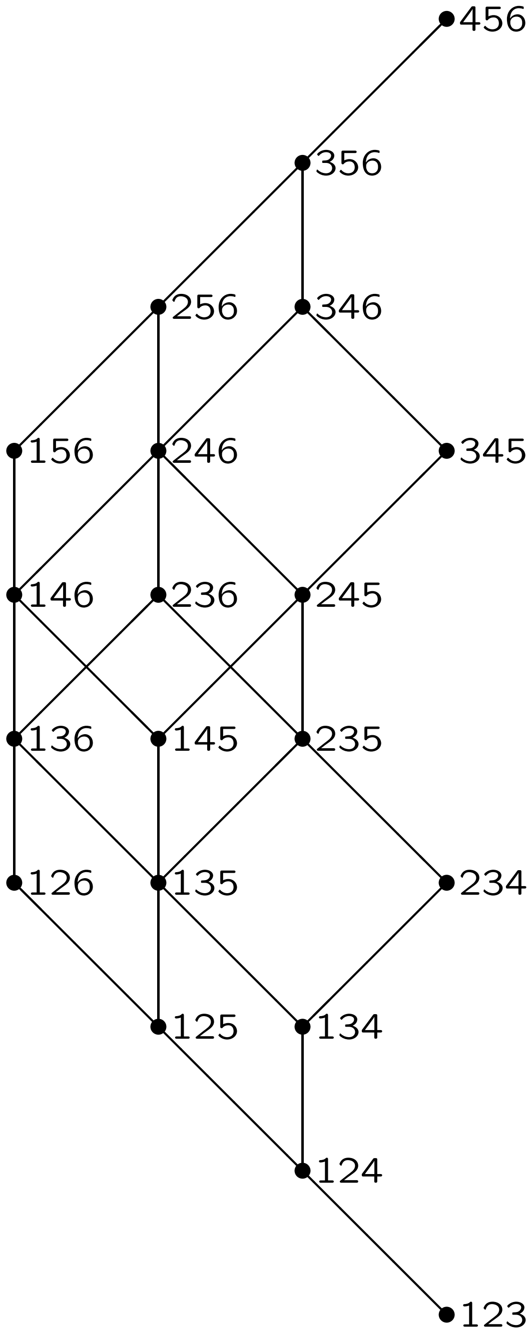
$$(F - v) \cup v' \in \mathcal{K}.$$

Example: 123, 124, 125, 126, 134, 135, 136, 145, 234, 235, 236.

A simplicial complex is shifted if its family of i-dimensional faces is shifted, for all i.

The simplicial complex formed by taking all subsets of every set $F \in \mathcal{K}$ is a pure shifted simplicial complex.

ORDER IDEAL


456
356
256
346
156
246
345
146
236
245
136
145
235
126
135
234
125
134
124
123

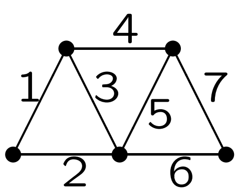
MATROIDS

Bases $\mathcal{B}$: non-empty family of k-subsets of ground set $E = \{1, \ldots, n\}$ satisfying

$\forall B \in \mathcal{B},\ \forall b \in B,\ \forall B' \in \mathcal{B},\ \exists b' \in B'$ such that

$$(B - b) \cup b' \in \mathcal{B}.$$

Example:



$\mathcal{B} =$

$(3 \in B)$		$(3 \notin B)$		
1346	2346	1246	1456	1467
1347	2347	1247	1457	2467
1356	2356	1256	2456	
1357	2357	1257	2457	
1367	2367	1267		

The simplicial complex formed by taking all subsets of every base $B \in \mathcal{B}$ is the set of independent sets IN(M) of matroid M.

RELATIVE PAIRS OF COMPLEXES

If $\Delta' \subseteq \Delta$ are a simplicial complexes on the same set of vertices, then $\Phi = (\Delta, \Delta') := \Delta - \Delta'$ is a relative pair of complexes.

When $\Delta' = \emptyset$, then $\Phi = (\Delta, \emptyset) = \Delta$.

Φ is an interval in the Boolean algebra.

LAPLACIANS

$C_i = C\Phi_i$, the i-dimensional $\mathbb{R}$-chains of Φ
($\mathbb{R}$-linear combinations of i-dim'l faces of Φ)

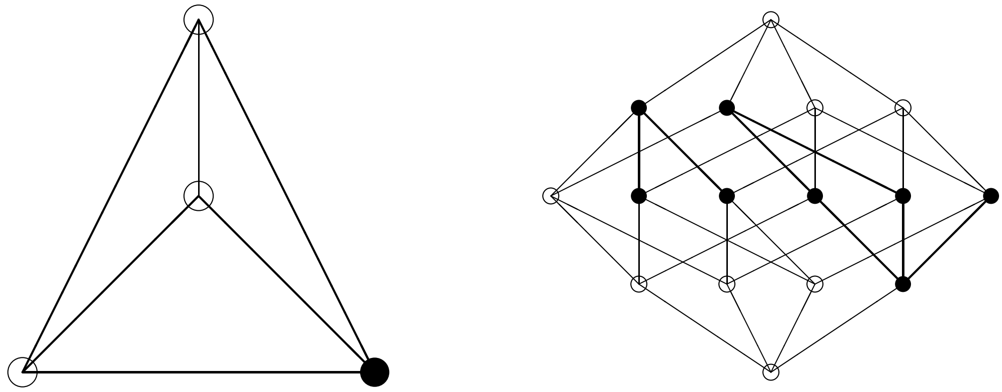
$\partial = \partial_i \colon C_i \to C_{i-1}$ usual signed boundary
$\delta_{i-1} = \partial_i^* \colon C_{i-1} \to C_i$ coboundary.

$$C_{i+1} \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_i \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_{i-1}$$

Defn: i-dimensional **Laplacian** of Φ:

$$L_i(\Phi) = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i \colon C_i \to C_i$$

Example:



EIGENVALUES OF LAPLACIANS

Easy observations about $\mathbf{s}$, eigenvalues

$\mathbf{s}(L_i) = \mathbf{s}(\partial_{i+1}\partial^*_{i+1}) \cup \ \mathbf{s}(\partial^*_i\partial_i)$

$\mathbf{s}(\partial^*_i\partial_i) = \mathbf{s}(\partial_i\partial^*_i)$, except for 0's

number of 0 eigenvalues is ith Betti number.

So we may as well just consider $s''_i = \mathbf{s}(\partial^*_i\partial_i)$; when $\Phi = (\Delta, \emptyset) = \Delta$, s''_i only depends on C_i.

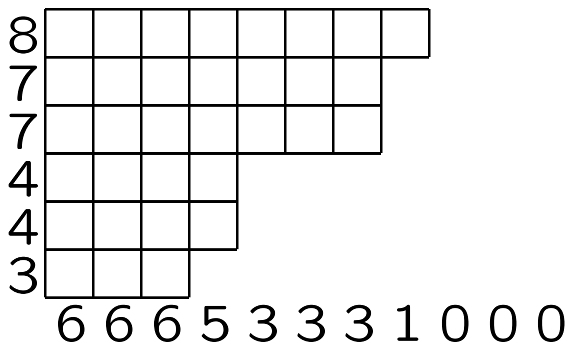
C_{i+1} $\mathbf{s}_{i+1}$	$\underset{\partial^*}{\overset{\partial}{\rightleftharpoons}}$	C_i $\mathbf{s}_i$	$\underset{\partial^*}{\overset{\partial}{\rightleftharpoons}}$	C_{i-1} $\mathbf{s}_{i-1}$
s''_{i+1}		s''_i		s''_{i-1}
$0^{\beta_{i+1}}$		0^{β_i}		$0^{\beta_{i-1}}$
s'_{i+1}		s'_i		s'_{i-1}

EIGENVALUES OF SHIFTED COMPLEXES

Defn d_i is the i-**dimensional degree sequence** $(d_i)_j = \#$ i-faces containing vertex j.

Example

123	134	234
124	135	235
125	136	236
126	145	



Thm (D-Reiner '02): If a simplicial complex is shifted, then

$$s''_i = (d_i)^T,$$

in every dimension i.

EIGENVALUES OF SHIFTED PAIRS

Defn: Assume $\mathcal{K}$ is a k-family, $\mathcal{K}'$ is a $(k-1)$-family, and $\mathcal{K}' \subseteq \partial\mathcal{K}$. Then

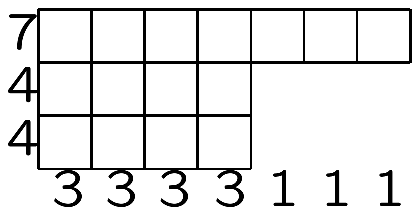
$$d_j(\mathcal{K}, \mathcal{K}') = \{F \in \mathcal{K} \colon F - j \notin \mathcal{K}'\},$$

and $d(\mathcal{K}, \mathcal{K}') = (d_1, \ldots, d_n)$.

$\mathcal{K}$	123	124	234	134	145	125	235	135	126	236	136
$\overline{\mathcal{K}'}$		24		34	45	25		35	26	16	36

$\mathcal{K}' = \{12, 13, 14, 15, 23\}$

Thm (D '03): If $\mathcal{K}$ and $\mathcal{K}'$ are shifted with the same vertex ordering, then $s(\mathcal{K}, \mathcal{K}') = d(\mathcal{K}, \mathcal{K}')^T$



EIGENVALUES OF MATROIDS

For matroids, eigenvalues are more easily described in terms of natural generating function:

$$S_M(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\mathsf{IN}(M)))} q^\lambda$$

Thm (Kook-Reiner-Stanton '00): For a matroid M with ground set E,

$$S_M(t,q) = q^{|E|} \sum_{I \in \mathsf{IN}(M)} t^{\mathsf{rank}(\bar{I})} (q^{-1})^{|\bar{\pi}(I)|},$$

where $\bar{\pi}(I)$ is a function of I involving internal/external activity.

(Ask about details later.)

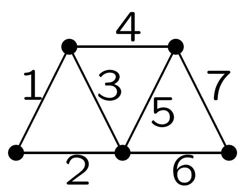
In particular, the eigenvalues of M are integers.

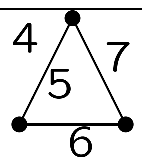
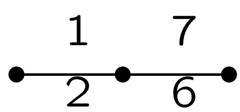
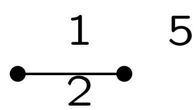
EIGENVALUES OF MATROID PAIRS

If the equivalent of pairs of shifted families with the same vertex ordering is *strong maps*, then it turns out we may as well restrict to $(M - e, M/e)$.

Removing M/e from $M - e$ partitions $M - e$, by the basic circuit ci(B, e), the unique circuit (minimal dependent set) in $B \cup e$, so

$$L(M - e, M/e) = \bigoplus_{\substack{C \text{ circuit} \\ e \in C}} L(M/C).$$



$C =$ ci(B, e)	123	345	3467
$M - e$	1246 1257 1247 1267 1256	1456 2456 1457 2457	1467 2467
M/C			

SPECTRAL RECURSION FOR MATROIDS...

Tutte polyn. deletion-contraction recursion:

$$T_M = T_{M-e} + T_{M/e}$$

$\mathcal{B}(M-e) = \{B \in \mathcal{B} \colon e \notin B\}$ $\quad (r = r(M))$

$\mathcal{B}(M/e) = \{B - e \colon B \in \mathcal{B},\ e \in B\}$ $\quad (r = r(M) - 1)$

Thm (Kook): $S_M = qS_{M-e} + qtS_{M/e}$
$+(1-q)$(error term).

Conj(Kook-Reiner): error term $= S_{(M-e,M/e)}$,
where $(M-e, M/e) = (\mathrm{IN}(M-e), \mathrm{IN}(M/e))$.

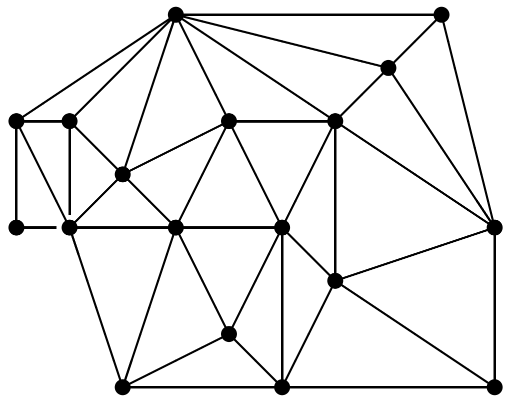
Thm (D '03): This is true, *i.e.*,

$$S_M = qS_{M-e} + qtS_{M/e} + (1-q)S_{(M-e,M/e)}.$$

. . . AND FOR SHIFTED COMPLEXES

Generalize deletion and contraction to arbitrary simplicial complex Δ.

$$\Delta - e = \{F \in \Delta \colon e \notin F\}$$
$$\Delta / e = \{F - e \colon F \in \Delta,\ e \in F\} \quad = \mathrm{lk}_\Delta\, e$$



$$S_\Delta(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Delta))} q^\lambda$$

Thm (D '03): Spectral recursion holds for shifted complexes Δ:

$$S_\Delta = qS_{\Delta - e} + qtS_{\Delta/e} + (1-q)S_{(\Delta-e,\Delta/e)}.$$

RELATIVE RECURSION

Say $\Phi = (\Delta, \Delta')$. Define

$$\begin{aligned}
\Phi - e &= \{F \in \Phi \colon e \notin F\} \\
&= (\Delta - e, \Delta' - e) \\
\Phi / e &= \{F - e \colon F \in \Phi,\ e \in F\} \\
&= (\Delta / e, \Delta' / e) \\
\Phi \parallel e &= \Phi - \{(F, F \dot\cup e) \colon (F, F \dot\cup e) \subseteq \Phi\} \\
&\approx (\Delta - e, (\Delta' - e \cup \Delta / e)) \\
&\quad \dot\cup ((\Delta' - e \cap \Delta / e), \Delta' / e)
\end{aligned}$$

Conj: If Δ, Δ' shifted w/same vertex order or matroids w/strong map, then

$$S_\Phi = q S_{\Phi - e} + qt S_{\Phi / e} + (1 - q) S_{\Phi \parallel e}$$

Example:

124 125 134 234

15 25 34 24 35

5

MORE ABOUT $\Phi \parallel e$

Original description of $(\Delta - e, \Delta/e)$ was $(\Delta, \mathrm{st}_\Delta\, e)$ (they are the same). In some sense, $\Phi \parallel e$ is $(\Phi, \mathrm{st}_\Phi\, e)$.

When plugging in $q = 0$, S is generating function for homology Betti numbers. $(\Delta, \mathrm{st}_\Delta\, e)$ has same homology as Δ, since $\mathrm{st}_\Delta\, e$ is contractible, so recursion for Δ is trivially true for all Δ. Same is true for Φ; note $\mathrm{st}_\Phi\, e = (\mathrm{st}_\Delta\, e, \mathrm{st}_{\Delta'}\, e)$.

EIGENVALUES OF MATROIDS
(details)

$$S_M(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\mathsf{IN}(M)))} q^\lambda$$

Thm (KRS '00): For matroid $M(E)$,

$$S_M(t,q) = q^{|E|} \sum_{I \in \mathsf{IN}(M)} t^{\mathsf{rank}(\bar{I})} (q^{-1})^{|\bar{\pi}(I)|},$$

where:

$I = \pi(I) \mathbin{\dot\cup} \sigma(I)$;

$\pi(I)$ has internal activity 0 in $\bar{I}$;

$\bar{\pi}(I) = \overline{\pi(I)}$; and

$\sigma(I)$ has external activity 0 in $\bar{I}/\bar{\pi}(I)$.

Etienne-Las Vergnas ('98) first showed that there is a unique such decomposition of I; the algorithm, due to KRS, to find this decomposition was essential to the proof of the spectral recursion for matroids.