# The surprising similarity of shifted simplicial complexes and matroids

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# Summary

Shifted simplicial complexes and (the independence complexes) of matroids have many similarities:

- Definitions are somewhat similar.
- Both closed under deletion and contraction.
- Eigenvalues of combinatorial Laplacians are integers (pretty rare).
- Eigenvalues satisfy same recursion (different proofs).

## Question

*Is there a (nice) common generalization of shifted complexes and matroids?*

# Shifted simplicial complexes

**Definition**

A non-empty family $\mathcal{K}$ of $k$-subsets of ground set $E = \{1, \ldots, n\}$ is shifted if: $\forall F \in \mathcal{K},\ \forall v \in F, \forall v' < v,$ if $v' \notin F,$ then

$$(F - v) \cup v' \in \mathcal{K}.$$

**Example**

123, 124, 125, 126, 134, 135, 136, 145, 234, 235, 236.

**Definition**

A simplicial complex is shifted if its family of $i$-dimensional faces is shifted, for all $i$.

**Remark**

The simplicial complex formed by taking all subsets of every set $F \in \mathcal{K}$ is a pure shifted simplicial complex.

# Independence complexes of matroids

## Definition

Matroid can be defined by its bases: A non-empty family $\mathcal{B}$ of $k$-subsets of ground set $E = \{1, \ldots, n\}$ satisfying:
$\forall B \in \mathcal{B},\ \forall b \in B, \forall B' \in \mathcal{B},\ \exists b' \in B'$ such that

$$(B - b) \cup b' \in \mathcal{B}.$$

## Example

If $G$ is a graph, then the bases of $M(G)$ are spanning trees.

## Definition

The independence complex $\mathrm{IN}(M)$ of $M$ is the simplicial complex formed by taking all subsets of every base $B \in \mathcal{B}$, i.e., the independent sets $\mathrm{IN}(M)$ of matroid $M$.

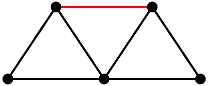
# Deletion and contraction

Motivated by the independent sets of a graph after deleting, or contracting, an edge of the graph.

## Definition

$$\mathrm{IN}(M - e) = \{I \in \mathrm{IN}(M) \colon e \notin I\}$$
$$\mathrm{IN}(M/e) = \{I - e \colon I \in \mathrm{IN}(M),\ e \in I\}$$

## Example



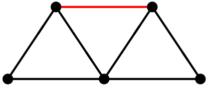
# Deletion and contraction

Motivated by the independent sets of a graph after deleting, or contracting, an edge of the graph. But it can also be done for any simplicial complex.

## Definition

$$\Delta - e = \{F \in \Delta \colon e \notin F\}$$
$$\Delta / e = \{F - e \colon F \in \Delta,\ e \in F\}$$

## Example



124, 125, 126, 145
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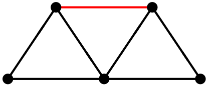
# Tutte recursion



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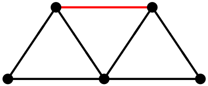
## Fact (easy)

*Matroids, and shifted complexes, are closed under deletion and contraction.*

# Tutte recursion



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## Fact (easy)

*Matroids, and shifted complexes, are closed under deletion and contraction.*

## Remark

Tutte polynomial satisfies:

$$T_M = T_{M-e} + T_{M/e}$$

and many matroid invariants are evaluations of the Tutte polynomial

# Laplacians

## Definition

- $L_i^{du} = \partial_i^T \partial_i : C_i \to C_i$, down-up Laplacian
- $L_i^{ud} = \partial_{i+1} \partial_{i+1}^T : C_i \to C_i$, up-down Laplacian
- $L_i^{tot} = L_i^{du} + L_i^{ud} : C_i \to C_i$, total Laplacian

where $\partial_i : C_i(\Delta) \to C_{i-1}(\Delta)$ is the usual signed boundary map.

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## Remark

Can get eigenvalues (in all dimensions) of any one of these from any other of them (basic linear algebra)

# Eigenvalues

**Theorem (D.-Reiner, '02; Kook-Reiner-Stanton, '00)**

*Laplacian eigenvalues of shifted complexes, and matroids, are integers, and there are nice formulas*

**Remark**

Very few other examples of integer Laplacian eigenvalues.

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**Remark**

We can also do all this for relative complexes (shifted complexes: same vertex ordering; matroids: strong map), e.g., $(\Delta - e, \Delta/e)$.

# Spectral recursion

**Definition**

Spectral polynomial $S_\Delta(q, t)$ is a generating function of Laplacian eigenvalues of a simplicial complex $\Delta$.

**Theorem (D., '05)**

*Both shifted complexes, and matroids, satisfy the spectral recursion:*

$$S_\Delta = qS_{\Delta-e} + qtS_{\Delta/e} + (1-q)S_{(\Delta-e,\Delta/e)}$$

**Remark**

Proof for shifted complexes totally different from proof for matroids.

# Weighted version?

## Question

*How much of this setup works with the following weighted boundary matrix?*

| | 123 | 124 | 125 | 134 | 234 |
|---|---|---|---|---|---|
| 12 | +3 | +4 | +5 | 0 | 0 |
| 13 | -2 | 0 | 0 | +4 | 0 |
| 14 | 0 | -2 | 0 | -3 | 0 |
| 15 | 0 | 0 | -2 | 0 | 0 |
| 23 | +1 | 0 | 0 | 0 | +4 |
| 24 | 0 | +1 | 0 | 0 | -3 |
| 25 | 0 | 0 | +1 | 0 | 0 |
| 34 | 0 | 0 | 0 | +1 | +2 |

## Remark (D.-Klivans-Martin, '09)

Weighted Laplacian eigenvalues of shifted complexes are nice.

# Summary

### Question

*What else satisfies all of these?:*

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- Integer Laplacian eigenvalues
- Satisfy spectral recursion

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Happy (birth+1)day, Richard!