Matroid Steiner complexes are Laplacian integral

AMS Regional Meeting

Bard College
Oct., '05

Matroid Steiner complexes are Laplacian integral

Art Duval,
University of Texas at El Paso

OVERVIEW

The **eigenvalues** of the **combinatorial Laplacian** of the independence complexes of **matroids** are **integral** and satisfy a Tutte-like **recursion**.

These things are true for very few other simplicial complexes. Some natural operations, including **Alexander duality**, preserve being integral and satisfying recursion. But Alexander dual of a matroid is not a matroid!

Ed Swartz and Federico Ardila say: **Steiner complexes** generalize matroids, and are closed under Alexander duality.

Theorem: Laplacian eigenvalues of Steiner complexes are integral. . .

Conjecture:. . . and satisfy the recursion.

MATROIDS
(Examples for graphic matroids)

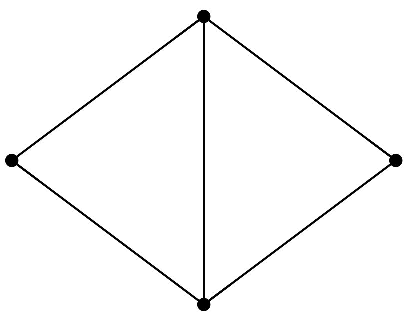
Ground set E: Edges of planar graph G.

Bases $\mathcal{B}$: Spanning trees of G. (Maximal indpendent sets.)

Independent sets $\mathcal{I}$: Forests of G. (Subsets of bases.)

For all matroids, not just graphic: Independence complex IN(M) of matroid M is simplicial complex of independent sets. (Facets are bases.)

Dual M^*: Planar graph dual.



LAPLACIANS

$C_i = C\Delta_i$, the i-dimensional $\mathbb{R}$-chains of Δ ($\mathbb{R}$-linear combinations of i-dim'l faces of Δ)

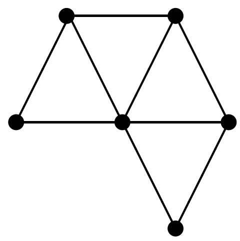
$\partial = \partial_i \colon C_i \to C_{i-1}$ usual signed boundary
$\delta_{i-1} = \partial_i^* \colon C_{i-1} \to C_i$ coboundary.

$$C_{i+1} \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_i \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_{i-1}$$

Let

$$L_i(\Delta) = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i \colon C_i \to C_i$$

be the i-dimensional Laplacian of Δ.



EIGENVALUES OF LAPLACIANS

$\mathbf{s}_i(\Delta) =$ eigenvalues (w/multiplicity) of $L_i(\Delta)$.

$\mathbf{s}$ integral for

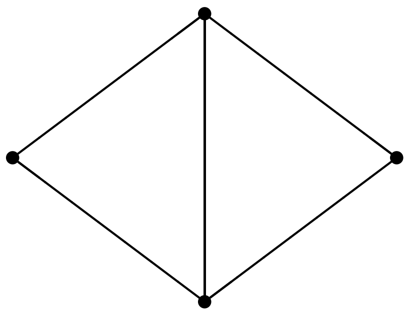
- independence complex IN(M) of matroid M (Kook-Reiner-Stanton, J. AMS '00)
- shifted complexes (D-Reiner, Trans. AMS '02)
- chessboard complexes (Friedman-Hanlon, J. Alg. Comb. '98)
- matching complexes of K_n (Dong-Wachs, Elec. J. Comb. '02)
- What else??!

SPECTRAL RECURSION

$$S_\Delta(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Delta))} q^\lambda$$

Tutte polyn. deletion-contraction recursion:

$$T_M = T_{M-e} + T_{M/e}$$



$$\mathcal{B}(M-e) = \{B \in \mathcal{B}\colon e \notin B\}$$
$$\mathcal{B}(M/e) = \{B - e\colon B \in \mathcal{B},\ e \in B\}$$

Spectral recursion:

$$S_\Delta = qS_{\Delta-e} + qtS_{\Delta/e} + (1-q)S_{(\Delta-e,\Delta/e)}$$

True for

- Matroids: Kook '04 (w/different error term); D '05
- Shifted complexes: D '05

ALEXANDER DUAL

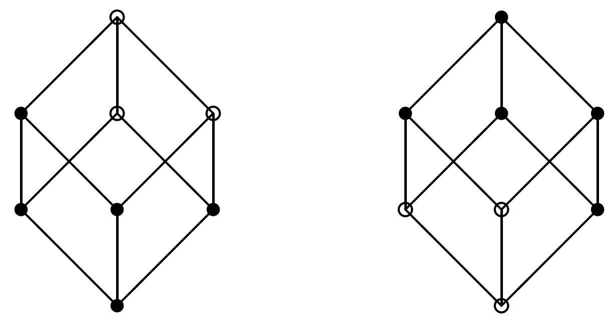
What else has integral Laplacian spectrum, and satisfies the spectral recursion? Call such complexes integral, and spectral, respectively.

One clue comes from duality: For matroids and Tutte polynomial, $T_{M^*}(x,y) = T_M(y,x)$, where $\mathcal{B}^* = \{E - B \colon B \in \mathcal{B}\}$.

Defns: **dual** $\Delta^* := \{E - F \colon F \in \Delta\}$

complement $\Delta^c := \{F \subseteq E \colon F \notin \Delta\}$

Alexander dual $\Delta^\vee := \Delta^{*c} = \Delta^{c*}$
$= \{E - F \colon F \notin \Delta\}$



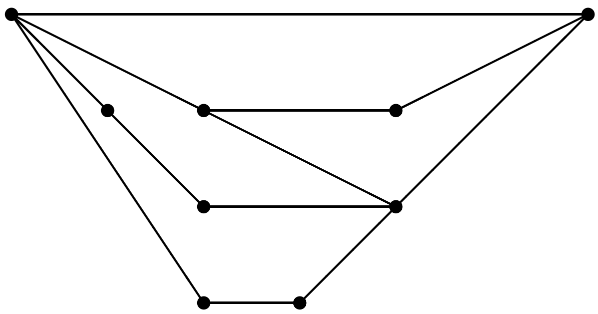
Thm (D '05): Δ integral (resp., spectral) iff $\Delta^\vee$ integral (resp., spectral).

STEINER COMPLEXES

circuits $\mathcal{C}(M)$, minimally dependent sets.

cocircuits $\mathcal{C}^*(M) = \mathcal{C}(M^*)$. (In graphic matroids, "cutsets".)

port $\mathcal{P}(M, e) = \{C - \{e\} \colon e \in C, C \in \mathcal{C}(M)\}$
$\mathcal{P}^*(M, e) = \{C^* - \{e\} \colon e \in C^*, C^* \in \mathcal{C}^*(M)\}$



Steiner complex (char'n of Chari '93)

$$\mathcal{S}(M, e) = \{F \subseteq E - \{e\} \colon P \not\subseteq F, \forall P \in \mathcal{P}\}$$

Generalizes matroids: $\mathcal{S}(M \times e, e) = \mathrm{IN}(M)$, where $\times$ denotes free coextension

DUALITY, etc.

Steiner complexes closed under deletion, contraction, Alexander duality:

$$\mathcal{S}(M,e) - f = \mathcal{S}(M-f,e)$$
$$\mathcal{S}(M,e)/f = \mathcal{S}(M/f,e)$$
$$\mathcal{S}(M,e)^{\vee} = \mathcal{S}(M^*,e)$$

From original defn (Colbourn-Pulleyblank '89), $\mathcal{P}$ and $\mathcal{P}^*$ are *blocking clutters*; each clutter (anti-chain) is minimal for intersecting each of the sets in the other clutter.

$$\mathcal{P}(M^*,e) = \mathcal{P}^*(M,e)$$

Thm: Steiner complexes are Laplacian integral . . .

Conj: . . . and satisfy spectral recursion.

RELATIVE PAIRS

Thm (D '05): Relative pairs of shifted complexes ($\Delta' \subseteq \Delta$, both shifted on same underlying vertex order) satisfy

$$S_\Phi = qS_{\Phi - e} + qtS_{\Phi/e} + (1-q)S_{\Phi\|e}$$

where $\Phi = \Delta - \Delta'$, for suitable $\Phi \parallel e$.

What about matroids? $(M-e, M/e)$ is integral, with a nice formula for eigenvalues; Vic Reiner suggests looking at (N, N'), where $N \to N'$ is a strong map. Perhaps even more generally (and vaguely) (S, S') where $S' \subseteq S$ are both Steiner complexes on same underlying matroid.