Port complexes and the Laplacian spectral recursion

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OVERVIEW

The **eigenvalues** of the **combinatorial Laplacian** of the independence complexes of **matroids** are **integral** and satisfy a Tutte-like **recursion**.

These things are true for very few other simplicial complexes. Some natural operations, including **Alexander duality**, preserve being integral and satisfying recursion. But Alexander dual of a matroid is not a matroid!

Ed Swartz says: **Steiner complexes** generalize matroids, and are closed under Alexander duality.

Conjecture: Laplacian eigenvalues of Steiner complexes are integral, and satisfy the recursion.

MATROIDS
(Examples for graphic matroids)

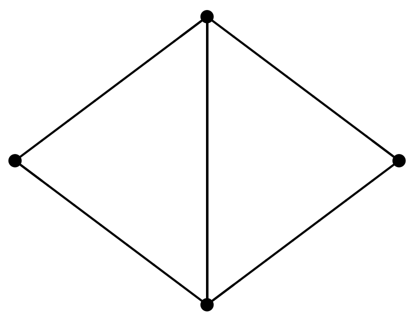
Ground set E: Edges of planar graph G.

Bases $\mathcal{B}$: Spanning trees of G. (Maximal indpendent sets.)

Independent sets $\mathcal{I}$: Forests of G. (Subsets of bases.)

For all matroids, not just graphic: Independence complex IN(M) of matroid M is simplicial complex of independent sets. (Facets are bases.)

Dual M^*: Planar graph dual.



LAPLACIANS

$C_i = C\Delta_i$, the i-dimensional $\mathbb{R}$-chains of Δ ($\mathbb{R}$-linear combinations of i-dim'l faces of Δ)

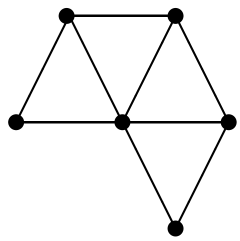
$\partial = \partial_i \colon C_i \to C_{i-1}$ usual signed boundary
$\delta_{i-1} = \partial_i^* \colon C_{i-1} \to C_i$ coboundary.

$$C_{i+1} \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_i \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_{i-1}$$

Let

$$L_i(\Delta) = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i \colon C_i \to C_i$$

be the i-dimensional Laplacian of Δ.



EIGENVALUES OF LAPLACIANS

$\mathbf{s}_i(\Delta)$ = eigenvalues (w/multiplicity) of $L_i(\Delta)$.

$\mathbf{s}$ integral for

- independence complex IN(M) of matroid M (Kook-Reiner-Stanton, J. AMS '00)
- shifted complexes (D-Reiner, Trans. AMS '02)
- chessboard complexes (Friedman-Hanlon, J. Alg. Comb. '98)
- matching complexes of K_n (Dong-Wachs, Elec. J. Comb. '02)
- What else??!

SPECTRAL RECURSION

$$S_M(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\mathsf{IN}(M)))} q^\lambda$$

Tutte polyn. deletion-contraction recursion:

$$T_M = T_{M-e} + T_{M/e}$$

$$\mathcal{B}(M-e) = \{B \in \mathcal{B} \colon e \notin B\} \qquad (r = r(M))$$
$$\mathcal{B}(M/e) = \{B - e \colon B \in \mathcal{B},\ e \in B\} \quad (r = r(M) - 1)$$

Thm (Kook): $S_M = qS_{M-e} + qtS_{M/e}$
$+(1-q)$(error term).

Conj(Kook-Reiner): error term $= S_{(M-e,M/e)}$, where $(M-e, M/e) = (\mathsf{IN}(M-e), \mathsf{IN}(M/e))$ is the "relative complex" of $\mathsf{IN}(M-e)$ with all the faces from $\mathsf{IN}(M/e)$ removed.

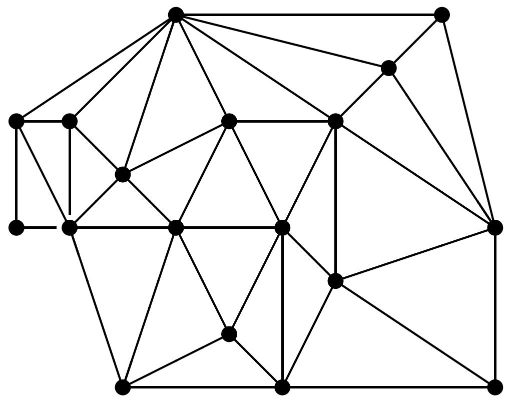
Thm: This is true, *i.e.*,

$$S_M = qS_{M-e} + qtS_{M/e} + (1-q)S_{(M-e,M/e)}.$$

MORE GENERALLY

Generalize deletion and contraction to arbitrary simplicial complex Δ.

$$\Delta - e = \{F \in \Delta \colon e \notin F\}$$
$$\Delta/e = \{F - e \colon F \in \Delta,\ e \in F\} \quad = \mathrm{lk}_\Delta\, e$$



$$S_\Delta(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Delta))} q^\lambda$$

Thm: Spectral recursion holds for shifted complexes Δ:

$$S_\Delta = qS_{\Delta-e} + qtS_{\Delta/e} + (1-q)S_{(\Delta-e,\Delta/e)}.$$

ALEXANDER DUAL

What else has integral Laplacian spectrum, and satisfies the spectral recursion? Call such complexes integral, and spectral, respectively.

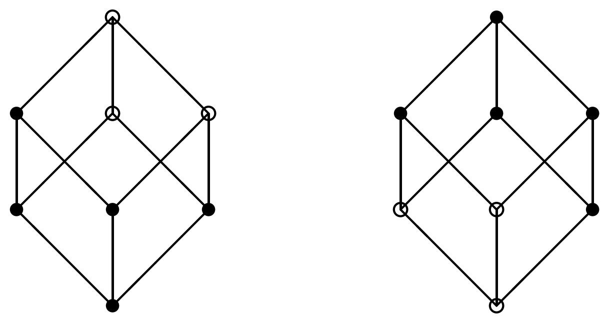
One clue comes from duality: For matroids and Tutte polynomial, $T_{M^*}(x,y) = T_M(y,x)$, where $\mathcal{B}^* = \{E - B \colon B \in \mathcal{B}\}$.

Defns:

$$\textbf{dual } \Delta^* := \{E - F \colon F \in \Delta\}$$

$$\textbf{complement } \Delta^c := \{F \subseteq E \colon F \notin \Delta\}$$

$$\textbf{Alexander dual } \Delta^\vee := \Delta^{*c} = \Delta^{c*} = \{E - F \colon F \notin \Delta\}$$



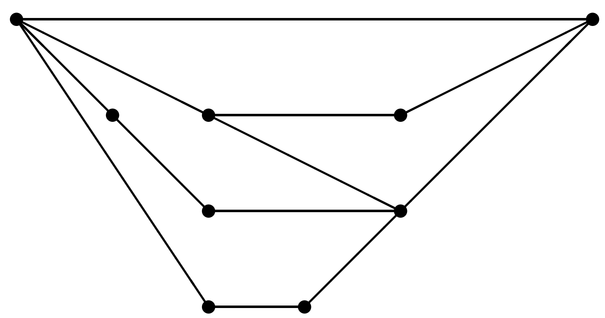
Thm: Δ integral (resp., spectral) iff $\Delta^\vee$ integral (resp., spectral).

STEINER COMPLEXES

circuits $\mathcal{C}(M)$, minimally dependent sets.

cocircuits $\mathcal{C}^*(M) = \mathcal{C}(M^*)$. (In graphic matroids, "cutsets".)

port $\mathcal{P}(M,e) = \{C - \{e\} \colon e \in C, C \in \mathcal{C}(M)\}$
$\mathcal{P}^*(M,e) = \{C^* - \{e\} \colon e \in C^*, C^* \in \mathcal{C}^*(M)\}$



Steiner complex

$$\mathcal{S}(M,e) = \{F \subseteq E - \{e\} \colon P \not\subseteq F, \forall P \in \mathcal{P}\}$$

Generalizes matroids: $\mathcal{S}(M \times e, e) = \mathrm{IN}(M)$, where $\times$ denotes free coextension

DUALITY, etc.

Steiner complexes closed under deletion, contraction, Alexander duality:

$$\mathcal{S}(M,e) - f = \mathcal{S}(M - f, e)$$
$$\mathcal{S}(M,e)/f = \mathcal{S}(M/f, e)$$
$$\mathcal{S}(M,e)^{\vee} = \mathcal{S}(M^*, e)$$

$\mathcal{P}$ and $\mathcal{P}^*$ are *blocking clutters*; each clutter (anti-chain) is minimal for intersecting each of the sets in the other clutter.

$$\mathcal{P}(M^*, e) = \mathcal{P}^*(M, e)$$

Conjecture: Steiner complexes are Laplacian integral and spectral.