Counting weighted simplicial spanning trees of shifted complexes

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MATRIX TREE THEOREM

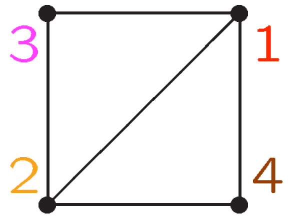
$$\sum_{T \in ST(G)} \text{wt}\, T = |\det \hat{L}_r(G)|,$$

where

$$\text{wt}\, T = \prod_{e \in T} \text{wt}\, e = \prod_{e \in T} \Big(\prod_{v \in e} x_v \Big),$$

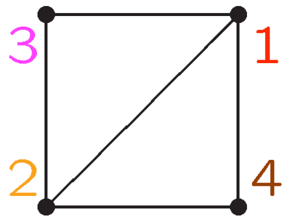
"Reduced": remove rows/columns corresponding to any one vertex

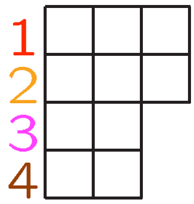
$\hat{L} = \partial \partial^T$, (weighted) Laplacian
∂, weighted boundary (incidence) matrix; the (e, v) entry is $\pm\sqrt{\text{wt}\, e}$

Example: 

$$\partial^T = \begin{pmatrix} -\sqrt{12} & \sqrt{12} & 0 & 0 \\ -\sqrt{13} & 0 & \sqrt{13} & 0 \\ -\sqrt{14} & 0 & 0 & \sqrt{14} \\ 0 & -\sqrt{23} & \sqrt{23} & 0 \\ 0 & -\sqrt{24} & 0 & \sqrt{24} \end{pmatrix}$$

EXAMPLE

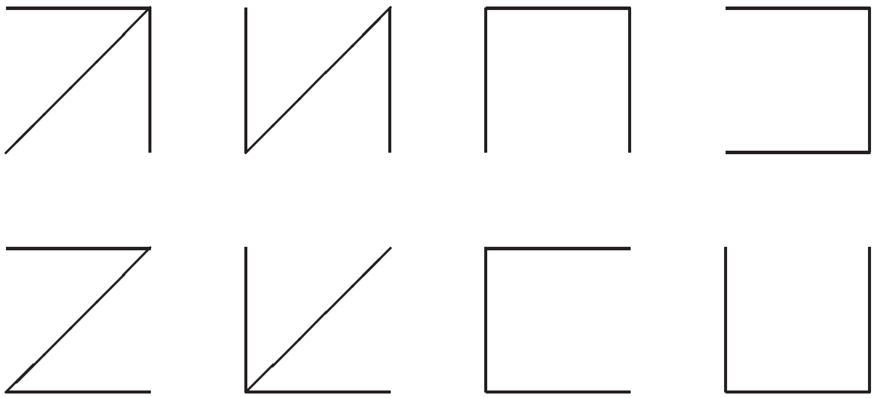




$$\hat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\hat{L}_r = \begin{pmatrix} 2(1+3+4) & -23 & -24 \\ -23 & 3(1+2) & 0 \\ -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\det \hat{L}_r = (1234)(1+2)(1+2+3+4)$$



SIMPLICIAL SPANNING TREES of COMPLETE SIMPLICIAL COMPLEX (Kalai '83)

Defn: Simplicial spanning tree: (Assume dim k; n vertices.) Set T, of k-dimensional faces, containing all $(k-1)$-dimensional faces and:

1. $|T| = \binom{n-1}{k}$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

How many are there?
Bolker ('76): should be $n^{\binom{n-2}{k}}$, but isn't

Kalai ('83): proves

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2 = n^{\binom{n-2}{k}}$$

WEIGHTING

$$\mathrm{wt}\, T = \prod_{F \in T} \mathrm{wt}\, F = \prod_{F \in T} \Big(\prod_{v \in F} x_v \Big)$$

Thm (Kalai '83):

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2 (\mathrm{wt}\, T)$$
$$= (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

Adin ('92) did something similar for complete r-partite complexes.

KALAI'S THEOREM

Proof (unweighted; weighted is similar):

$$\begin{aligned} n^{\binom{n-2}{k}} &= \det L_r(K_n^k) = \det \partial_r(K_n^k)\partial_r(K_n^k)^T \\ &= \sum_T (\det \partial_r(T))^2 \\ &= \sum_T |\tilde{H}_{k-1}(T)|^2 \end{aligned}$$

by Binet-Cauchy.

"Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

∂: $\Delta_k \to \Delta_{k-1}$ boundary
∂^T: $\Delta_{k-1} \to \Delta_k$ coboundary

SIMPLICIAL MATRIX TREE THEOREM

Defn: Simplicial spanning tree of Δ: (Assume $\dim \Delta = k$.) k-dimensional complex T containing all $(k-1)$-dimensional faces of Δ ($T^{(k-1)} = \Delta^{(k-1)}$) and:

1. $f_k(T) = f_k(\Delta) - \beta_k(\Delta) + \beta_{k-1}(\Delta)$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

Thm (DKM):

$$\sum_{T \in SST(\Delta)} |\tilde{H}_{k-1}(T)|^2 \operatorname{wt} T = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det \hat{L}_r.$$

U =set of facets of $(k-1)$-spanning tree of Δ
$\hat{L}_r$ is L reduced by all of U
$\Delta_U = U \cup \Delta^{(k-2)}$

SHIFTED SIMPLICIAL COMPLEXES

Defn: $V = 1, \ldots, n$
$F \in \Delta, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta$ (equivalently, the k-faces form an initial ideal in the componentwise partial order).

Ex: bipyramid = { 123, 124, 125, 134, 135, 234, 235} (and subfaces)

$$\Delta = (1 * \text{lk}_\Delta 1) \dot\cup B_\Delta$$

$B_\Delta = \{F \in \Delta \colon 1 \notin F, F \dot\cup 1 \notin \Delta\}$
$\text{lk}_\Delta 1 = \{F - 1 \colon 1 \in F, F \in \Delta\}$, shifted
$\text{del}_\Delta 1 = \{G \colon 1 \notin G, G \in \Delta\}$, shifted
$\beta_i(\Delta) = f_i(B_\Delta)$
$\text{del}_\Delta 1 = \text{lk}_\Delta 1 \dot\cup B_\Delta$

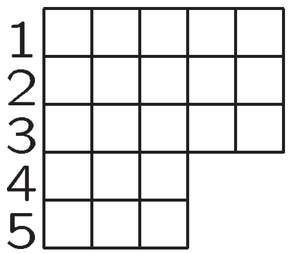
Example (bipyramid, continued):
$B_\Delta = \{234, 235\}$
$\text{lk}_\Delta 1 = \{23, 24, 25, 34, 25; 2, 3, 4, 5; \emptyset\}$
$\text{del}_\Delta 1 = \text{lk}_\Delta 1 \dot\cup B_\Delta$.

EIGENVALUES

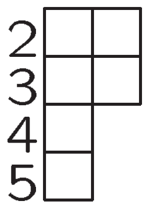
Thm (D-Reiner '02): Non-0 eigenvalues of top-dim'l Laplacian of shifted complex given by d^T where d is degree sequence, $d_i = |\{\text{facets } F : i \in F\}|$.

Example (bipyramid):

$\{123, 124, 125, 134, 135, 234, 235\}$



Example (del1): $234, 235$



COUNTING TREES OF SHIFTED COMPLEXES

Pick U to be ridges ($(k-1)$-dimensional faces) containing 1, which is acyclic, and contains $(k-2)$-faces of Δ, and so it is (the facets of) a simplicial spanning tree.

Also, $\tilde{H}_{k-2}(\Delta_U) = 0$, and if Δ is pure and shifted, then $\tilde{H}_{k-2}(\Delta) = 0$, so we just have to compute $\det \hat{L}_r$.

Ex: bipyramid. Set of all ridges is all possible edges, except 45. $U = \{12, 13, 14, 15\}$, so $\hat{L}_r$ is indexed by $\{23, 24, 25, 34, 35\}$.

$$\hat{L}_r =$$

$$\begin{array}{ccccc}
23(1+4+5) & -234 & -235 & 234 & 235 \\
-234 & 24(1+3) & 0 & -234 & 0 \\
-235 & 0 & 25(1+3) & 0 & -235 \\
234 & 234 & 0 & (1+2)34 & 0 \\
235 & 0 & -235 & 0 & (1+2)35
\end{array}$$

SIMPLIFICATIONS

$$\det \hat{L}_r = (23)(24)(25)(34)(35)\det M$$

$$\det M = \begin{vmatrix} 1+4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 1+3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 1+3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 1+2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 1+2 \end{vmatrix}$$

$$= \left| 1I + N \right|$$

$$N = \begin{pmatrix} 4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 2 \end{pmatrix}$$

Remarkably, N is a weighted Laplacian of $\text{del}_{\triangle}1$; $N = \partial\partial^T$ with

$$\partial^T = \begin{pmatrix} \sqrt{4} & -\sqrt{3} & 0 & \sqrt{2} & 0 \\ \sqrt{5} & 0 & -\sqrt{3} & 0 & \sqrt{2} \end{pmatrix}$$

The (F, G) entry of this matrix is $\pm\sqrt{F - G}$.

FURTHER SIMPLIFICATIONS

This makes the eigenvalues of this N be
0, 0, 0, $2+3$, $2+3+4+5$, and
$\det M = (1)^3(1+2+3)(1+2+3+4+5)$.
Finally it makes the weighted tree enumerator of the bipyramid

$$(23)(24)(25)(34)(35) \\ \times (1)^3(1+2+3)(1+2+3+4+5).$$

More generally,

$$\Big(\prod_{F\in R} x_F\Big) \prod_{j=1}^{|R|} \sum_{i=1}^{1+(d^T)_j} x_i,$$

where R = facets of $\mathrm{lk}_\Delta 1$ and d is the degree sequence of $\mathrm{del}_\Delta 1$.