Weighted enumeration of spanning trees of complete colorful complexes and skeletons of hypercubes

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Algebra seminar
Texas State University
March 1, 2018

Spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

$T \subseteq E(G)$ is a **spanning tree** of G when:

0. spanning: T contains all vertices;
1. connected $(\tilde{H}_0(T) = 0)$
2. no cycles $(\tilde{H}_1(T) = 0)$
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

Weighted enumeration

Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \mathrm{wt}\, T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\mathrm{wt}\, T = \prod_{e \in T} \mathrm{wt}\, e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

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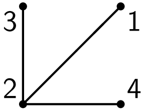
Example (K_4)

- 4 trees like: $T =$ 

$\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2$

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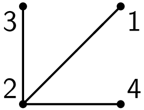
- 4 trees like: $T =$ 

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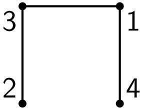
- 12 trees like: $T =$ 

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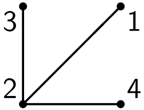
- 4 trees like: $T =$ 

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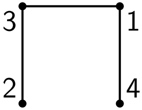
- 12 trees like: $T =$ 

$\text{wt } T = (x_1x_2x_3x_4)x_1x_3$
- Total is $(x_1x_2x_3x_4)(x_1 + x_2 + x_3 + x_4)^2$.

Complete bipartite graphs

Example ($K_{3,2}$)

Complete bipartite graphs

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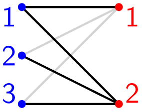
- 6 trees like: $T =$  wt $T = (12312)12^2$

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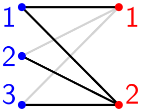
- 6 trees like: $T =$ 

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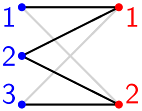
- 6 trees like: $T =$ 

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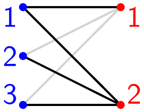
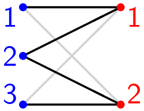
- 6 trees like: $T =$  wt $T = (12312)12^2$
- 6 trees like: $T =$  wt $T = (12312)212$
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Complete bipartite graphs

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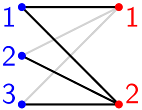

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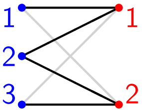

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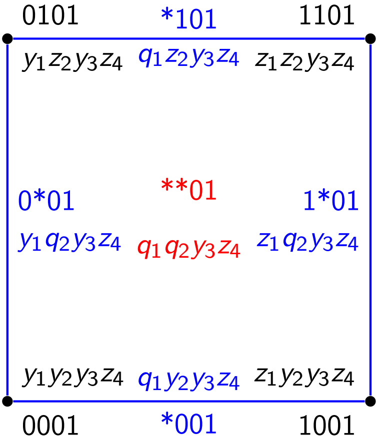
- Total is $(12312)(1+2+3)(1+2)^2$.

Theorem

$$\sum_{T \in ST(K_{m,n})} \text{wt } T = (x_1 \cdots x_m)(y_1 \cdots y_n)(x_1 + \cdots + x_m)^{n-1}(y_1 + \cdots + y_n)^{m-1}.$$

Cubical graphs

Example



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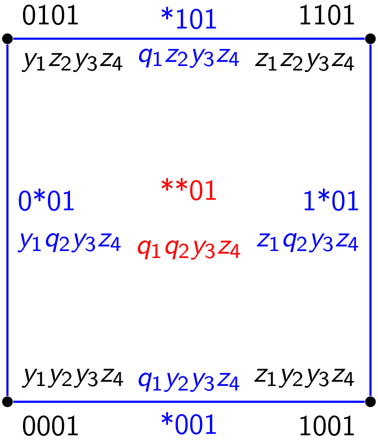
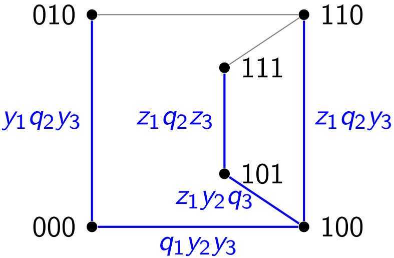




$$\text{wt } T = (y_1 q_2 y_3)(z_1 q_2 z_3)(z_1 y_2 q_3)(q_1 y_2 y_3)(z_1 q_2 y_3)$$

Enumerating trees of cubical graphs

Theorem (Martin-Reiner '03)

$$\sum_{T \in ST(Q_n)} \text{wt } T = q_{[n]} \prod_{\substack{S \subseteq [n] \\ |S| > 1}} (u_S y_S z_S)$$

where $u_S = \sum_{i \in S} q_i (\frac{1}{y_i} + \frac{1}{z_i})$

Example ($n = 3$)

$q_{[3]}(u_{12}y_{12}z_{12})(u_{13}y_{13}z_{13})(u_{23}y_{23}z_{23})(u_{123}y_{123}z_{123})$

Matrix-tree theorem (for graphs)

Let G be a graph with n vertices; let $\partial(G)$ is the oriented boundary matrix (oriented incidence matrix); let

$$L = \partial(G)\partial^T(G)$$

be the Laplacian of G.

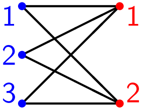
Theorem (Kirchoff's Matrix-Tree)

The number of spanning trees of graph G is

$$(\lambda_1\lambda_2\cdots\lambda_{n-1})/n$$

where $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of L.

Example: Complete bipartite graph



$\partial =$

	11	12	21	22	31	32
1	-1	-1	0	0	0	0
2	0	0	-1	-1	0	0
3	0	0	0	0	-1	-1
1	1	0	1	0	1	0
2	0	1	0	1	0	1

$$L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}$$

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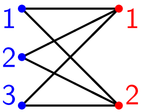

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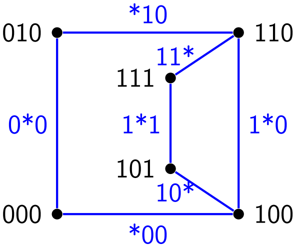
eigenvalues(L): 5, 3, 2, 2, 0; spanning trees: $(5 \cdot 3 \cdot 2 \cdot 2)/5 = 12$

Example: Cubical graph

$\partial =$

	*00	*10	0*0	1*0	1*1	10*	11*
000	+1	0	+1	0	0	0	0
010	0	+1	-1	0	0	0	0
100	-1	0	0	+1	0	+1	0
101	0	0	0	0	+1	-1	0
110	0	-1	0	-1	0	0	+1
111	0	0	0	0	-1	0	-1



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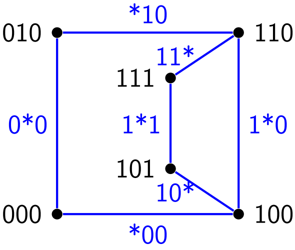

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

eigenvalues(L): 5,3,3,2,1,0; spanning trees: $(5 \cdot 3 \cdot 3 \cdot 2 \cdot 1)/6 = 15$

Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete d-dimensional simplicial complex on n vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^d)} \qquad = n^{\binom{n-2}{d}}$$

Counting simplicial spanning trees of K_n^d

Theorem [Kalai '83]

$$\tau(K_n^d) = \sum_{\Upsilon \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}$$

Weighted simplicial spanning trees of K_n^d

As before,

$$\operatorname{wt}\Upsilon = \prod_{F\in\Upsilon} \operatorname{wt} F = \prod_{F\in\Upsilon} \left(\prod_{v\in F} x_v\right)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$
$$\operatorname{wt}\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Weighted simplicial spanning trees of K_n^d

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Theorem (Kalai, '83)

$$\hat{\tau}(K_n^d) := \sum_{T \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 (\operatorname{wt} \Upsilon)$$
$$= (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}$$

Cellular spanning trees of arbitrary cell complexes

Let Δ be a d-dimensional cell complex.
$\Upsilon \subseteq \Delta$ is a **cellular spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_{d}(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Examples

Example (Octahedron)

- Vertices 1, 2, 1, 2, 1, 2.
- Facets 111, 112, 121, 122, 211, 212, 221, 222.

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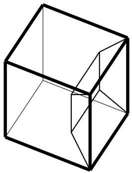

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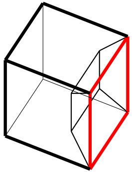

5×5 trees that don't include the red square; $5 + 5$ trees that do include the red square

Weighted Laplacian (arbitrary dimension)

weighted boundary map

$$\hat{\partial} = \begin{array}{c|ccc} & \cdots & \sigma & \cdots \\ \hline \vdots & \ddots & \vdots & \ddots \\ \rho & \cdots & \pm\sqrt{\sigma/\rho} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{array} \qquad \frac{\sigma}{\rho} = \frac{q_i}{x_i} \text{ or } \frac{q_i}{y_i}$$

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$\hat{\partial}_k = D_{k-1}^{-1} \partial_k D_k$, where $D_k = \mathrm{diag}(\ldots, \sqrt{\sigma_i^k}, \ldots)$.

(Note that this forms a chain complex: $\hat{\partial}_{k-1}\hat{\partial}_k = 0$.)

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weighted Laplacian

$$\hat{L}_{k-1} = \hat{\partial}_k \hat{\partial}_k^T = D_{k-1}^{-1}\partial_k D_k^2 \partial_k^T D_{k-1}^{-1}$$

Weighted Matrix-Tree Theorem

Recall $\hat{L}_{k-1} = \hat{\partial}_{k-1}\hat{\partial}_{k-1}^T$

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Let $X_{(k-1)} :=$ product of all faces of dimension $k-1$

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Enumerate spanning trees by
$\hat{\tau}_k := \sum_{\Upsilon \in ST(\Delta)} |\tilde{H}_{k-1}(\Upsilon)|^2 \, \mathrm{wt}(\Upsilon)$

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Theorem (ADKLM, slight genzn of MMRW)

$$\hat{\pi}_k = \frac{\hat{\tau}_k \hat{\tau}_{k-1}}{|\tilde{H}_{k-1}(\Delta)| X_{(k-1)}}$$

Complete colorful complexes

Definition (Adin, '92)

The complete colorful complex $K_{n_1,\ldots,n_r}$ is a simplicial complex with:

- vertex set $V_1 \dot\cup \ldots \dot\cup V_r$ (V_i is set of vertices of color i);
- $|V_i| = n_i$;
- faces are all sets of vertices with no repeated colors.

Example

Octahedron is K_{222}.

Unweighted enumeration

Theorem (Adin, '92)

The top-dimensional spanning trees of $K_{n_1,\ldots,n_r}$ are "counted" by

$$\tau(K_{n_1,\ldots,n_r}) = \prod_{i=1}^{r} n_i^{\prod_{j\neq i}(n_j-1)}.$$

Note: Adin also has a more general formula for dimension less than $r-1$.

Example

- $\tau(K_{222}) = 2^1 \times 2^1 \times 2^1$
- $\tau(K_{235}) = 2^{2\cdot 4} \times 3^{1\cdot 4} \times 5^{1\cdot 2}$
- $\tau(K_{m,n}) = m^{n-1} \times n^{m-1}$

Weighted enumeration

Theorem (ADKLM)

The top-dimensional spanning trees of $K_{n_1,\ldots,n_r}$ are "counted" by $\hat{\tau}(K_{n_1,\ldots,n_r}) =$

$$\prod_{i=1}^{r}(x_{i,1}+\cdots+x_{i,n_i})^{\prod_{j\neq i}(n_j-1)}(x_{i,1}\cdots x_{i,n_i})^{(\prod_{j\neq i}n_j)-(\prod_{j\neq i}(n_j-1))}.$$

Note: We also have a more general formula for dimension less than $r-1$.

Example

$$\begin{aligned}\hat{\tau}(K_{235}) &= (x_1+x_2)^{2\cdot 4}(x_1x_2)^{3\cdot 5-2\cdot 4}\\ &\times (y_1+y_2+y_3)^{1\cdot 4}(y_1y_2y_3)^{2\cdot 5-1\cdot 4}\\ &\times (z_1+\cdots+z_5)^{1\cdot 2}(z_1\cdots z_5)^{2\cdot 3-1\cdot 2}\end{aligned}$$

Weighted enumeration of spanning trees of $Q_{n,k}$

Conjecture (D-Klivans-M)

$$\hat{\tau}_k(Q_n) = q_{[n]}^{\sum_{i=k-1}^{n-1} \binom{n-1}{i}\binom{i-1}{k-2}} \prod_{\substack{S \subseteq [n] \\ |S| > k}} (u_S y_S z_S)^{\binom{|S|-2}{k-1}}$$

where $u_S = \sum_{i \in S} q_i(\frac{1}{y_i} + \frac{1}{z_i})$

Weighted enumeration of spanning trees of $Q_{n,k}$

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where $u_S = \sum_{i\in S} q_i(\frac{1}{y_i} + \frac{1}{z_i})$

Proof.

Relies on linear algebra tricks from $\hat{\partial}$ making chain complex, and eigenvalues of Q_n (DKM). □

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where $u_S = \sum_{i\in S} q_i(\frac{1}{y_i} + \frac{1}{z_i})$

Proof.
Relies on linear algebra tricks from $\hat{\partial}$ making chain complex, and eigenvalues of Q_n (DKM). □

Example

$\hat{\tau}_2(Q_4) = q_{[4]}^7 (u_{123}y_{123}z_{123})^1 \cdots (u_{234}y_{234}z_{234})^1 (u_{[4]}y_{[4]}z_{[4]})^2$

Another interpretation of the exponents

Theorem (ADKLM)

$$\hat{\tau}_k(Q_n) = q_{[n]}^{|\tilde{\chi}(Q_{n-1}^{(k-2)})|}(y_{[n]}z_{[n]})^{|\tilde{\chi}(Q_{n-1}^{(k-1)})|} \prod_{\substack{S\subseteq[n] \\ |S|>k}} u_S^{\binom{|S|-2}{k-1}}$$

where $\tilde{\chi}(\Delta)$ refers to the reduced Euler characteristic of a complex.