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LAPLACIAN EIGENVALUES OF BIPARTITE KNESER-LIKE GRAPHS

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ABSTRACT. Given $a, b \in \mathbb{N}$ such that $a > b$, $G(a, b)$ is a bipartite Kneser-like graph whose sets of vertices are a-sized and b-sized subsets of $S = \{1, .., a+b+1\}$, where the edges are formed if the subsets of two vertices are disjoint. We conjecture that all the eigenvalues of the Laplacian matrix of this graph are all non-negative integers. We prove that eigenvalue spectrum of the Laplacian matrix of $G(a, b)$ are symmetric and we describe this symmetry and its consequences. We also prove that $b = 2$ and $a + 2$ are eigenvalues of $L(G(a, 2))$. In addition to this, we conjecture a formula about the multiplicities of the eigenvalues.

1. Introduction

The eigenvalues of matrices are not often integers, rarely even rational numbers; however, it appears that the eigenvalues of the Laplacian matrix of these bipartite Kneser-like graphs are non-negative integers. We prove in section 3 that the spectrum of the eigenvalues and their associated multiplicities are symmetric across the $b+1$ eigenvalue. In addition to this, we prove that if our conjecture about the eigenvalues are true, then we have a general formula for the multiplicity, this is shown to be true in Theorem 5.2. In section 7 we prove that $b = 2$ and $a + 2$ are eigenvalues of $L(G(a, 2))$ with eigenvectors given by 2-dimensional signed cross-polytopes.

2. Notations and Definitions

Definition 2.1 (Construction of $G(a, b)$)**.** Given $a, b \in \mathbb{N}$ such that $a > b$, the graph $G(a, b)$ can be constructed in the following way:

Form $S = \{1, .., n\}$ where $n = a + b + 1$, from this the first set of vertices are all subsets of S such that they have a cardinality a, we denote these vertices as $\mathcal{A}$. Similarly the second set of vertices are all subsets of S such that they have a cardinality b, this set of vertices we denote as $\mathcal{B}$.

An edge is formed between a vertex $A \in \mathcal{A}$ and a vertex $B \in \mathcal{B}$, if and only if $A \cap B = \emptyset$.

The Laplacian Matrix of $G(a, b)$ is denoted by $L(G(a, b))$.

Definition 2.2 (Laplacian of a Matrix)**.** The Laplacian of a matrix is $D(M) - Adj(M)$, where $D(M)$ is the degree matrix of the graph M and $Adj(M)$ is the adjacency matrix of the graph M.

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The eigenvalue spectrum of $L(G(a,b))$ is denoted by $\sigma(G)$. $L(G(a,b))$ can be generalized as a block form matrix that is as follows:

$$\begin{bmatrix} (a+1)I & C \\ C^T & (b+1)I \end{bmatrix} \tag{2.1}$$

C is the matrix that transforms a-tuples into b-tuples and its transpose C^T is the matrix that transforms b-tuples into a-tuples. The dimensions of $(a+1)I$,C,C^T, and $(b+1)I$ are $\binom{n}{b} \times \binom{n}{b}$, $\binom{n}{b} \times \binom{n}{a}$, $\binom{n}{a} \times \binom{n}{b}$, and $\binom{n}{a} \times \binom{n}{a}$, respectively.

Definition 2.3 (Join Operator)**.** The join operator, denoted as $*$, is defined as $A * B = \{(a,b) : a \in A, b \in B\}$.

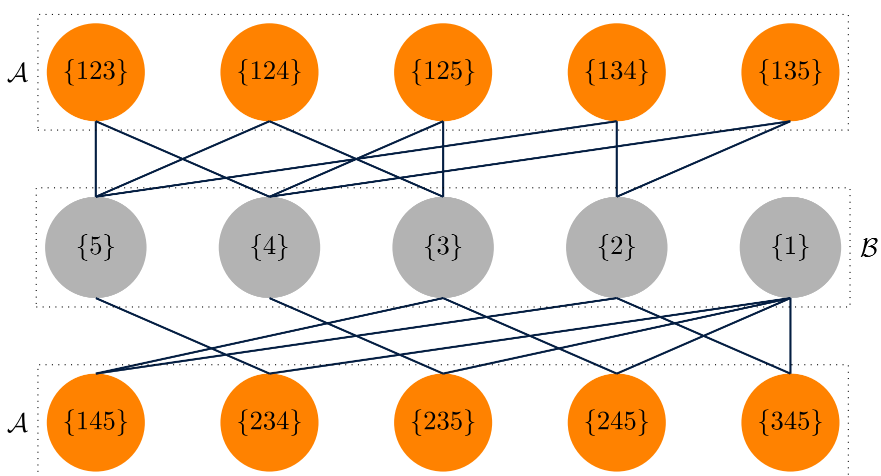
Example 2.4 (Formation of G(3,1))**.** Given $a = 3$ and $b = 1$ we can construct $G(3,1)$ in the way described in Definition 2.1. We can form our set $S = \{1,2,3,4,5\}$, from this we can form our set of vertices. The vertices in $\mathcal{A}$ are going to be all the subsets of S of length a, therefore we will have $\binom{5}{3} = 10$ vertices, so,

$$\mathcal{A} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\}, \\ \{2,3,4\},\{2,3,5\},\{3,4,5\}\}.$$

Similarly the vertices in $\mathcal{B}$ are going to be all the subsets of S of length b, therefore we will have $\binom{5}{1} = 5$ vertices, so,

$$\mathcal{B} = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$$

An edge is formed between a vertex in $\mathcal{A}$ and a vertex in $\mathcal{B}$ if and only if they are disjoint. This gives us $G(3,1)$,



The two partite sets, $\mathcal{A}$ and $\mathcal{B}$, are denoted by the color of the vertices, orange and silver respectively.

Example 2.5 (L(G(3,1)))**.** From the graph in Example 2.4, we can form the Laplacian Matrix, as described in Definition 2.2,

$$\left(\begin{array}{ccccc|cccccccccc}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 4 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 4 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right)$$

From this matrix it is clear to see where the block matrix from equation (2.1) came from, so the above matrix can be simplified into its specfic block form matrix,

$$\begin{bmatrix} 4I & C \\ C^T & 2I \end{bmatrix}.$$

3. Previous Work and Theorems

Cesar Vazquez proved these theorems in his Masters thesis, we include the theorems that will be useful for the proofs later on.

Theorem 3.1. *Given $a, b \in \mathbb{N}$ where $a > b$, the number of vertices of $G(a, b)$ is* $\binom{n}{a} + \binom{n}{b} = \binom{n+1}{b+1} = \binom{n+1}{a+1}$.

Theorem 3.2. $\lambda_1(G) = 0$ *is a simple eigenvalue of $L(G(a, b))$ and its corresponding eigenvector is* $(1, ..., 1)^T = \mathbf{j}$.

Theorem 3.3. $a + b + 2$ *is an eigenvalue of $L(G(a, b))$ with corresponding eigenvector $[(a + 1)\mathbf{j}, (-b - 1)\mathbf{j}]^T$ where the first $\binom{n}{b}$ components of the eigenvector are $(a + 1)$ and the last $\binom{n}{a}$ components are $(-b - 1)$.*

Theorem 3.4. $b + 1$ *is an eigenvalue of $L(G(a, b))$ with multiplicity* $\binom{n}{a} - \binom{n}{b}$.

These theorems are a few from Cesar's thesis that were proved for the general case of $L(G(a, b))$ as opposed to theorems just for $L(G(a, 1))$.

4. Symmetry Across Eigenvalue Spectrum

We start this section with the first theorem that shows this symmetry:

Theorem 4.1. *If $[x, y]$ is an eigenvector of $L(G(a, b))$ with an eigenvalue λ, then $[x, -\frac{b+1-\lambda}{a+1-\lambda}y]$ is an eigenvector of $L(G(a, b))$ with an eigenvalue $\phi = a + b + 2 - \lambda$, such that $a \neq \lambda - 1$.*

Proof. Assume that $[x, y]$ is a block eigenvector of $L(G(a, b))$ with an eigenvalue λ, thus,

$$\begin{bmatrix}(a+1)I & C \\ C^T & (b+1)I\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \lambda\begin{bmatrix}x\\y\end{bmatrix} \tag{4.1}$$

So, from (4.1), we can create the following system of equations,

$$\begin{cases}(a+1)x + Cy = \lambda x \\ C^T x + (b+1)y = \lambda y\end{cases} = \begin{cases}Cy = (\lambda - (a+1))x \\ C^T x = (\lambda - (b+1))y\end{cases} \tag{4.2}$$

Now for some modulated vector $[x, -\frac{b+1-\lambda}{a+1-\lambda}y]$,

$$\begin{bmatrix}(a+1)I & C \\ C^T & (b+1)I\end{bmatrix}\begin{bmatrix}x\\ -\frac{b+1-\lambda}{a+1-\lambda}y\end{bmatrix} = \begin{bmatrix}(a+1)x - \frac{b+1-\lambda}{a+1-\lambda}Cy \\ C^T x - \frac{b+1-\lambda}{a+1-\lambda}(b+1)y\end{bmatrix} \tag{4.3}$$

Thus from (4.2), we can perform the following substitution,

$$\begin{bmatrix}(a+1)x - \frac{b+1-\lambda}{a+1-\lambda}(\lambda-(a+1))x \\ (\lambda-(b+1))y - \frac{b+1-\lambda}{a+1-\lambda}(b+1)y\end{bmatrix} = \begin{bmatrix}x((a+1)+(b+1-\lambda)) \\ y((\lambda-(b+1)) + \frac{b+1-\lambda}{a+1-\lambda}(b+1))\end{bmatrix} = \tag{4.4}$$

$$\begin{bmatrix}x(a+b+2-\lambda) \\ y[(\lambda-(b+1)) + \frac{b+1-\lambda}{a+1-\lambda}(b+1)]\end{bmatrix} = \begin{bmatrix}\phi x \\ y[\lambda - (b+1)(1 + \frac{b+1-\lambda}{a+1-\lambda})\end{bmatrix} =$$

$$\begin{bmatrix}\phi x \\ y(\frac{\lambda(a+1-\lambda)-(b+1)(a+b+2-2\lambda)}{a+1-\lambda})\end{bmatrix} = \begin{bmatrix}\phi x \\ y(\frac{a\lambda+3\lambda-\lambda^2-ba-b^2-3b+2b\lambda-a-2}{a+1-\lambda})\end{bmatrix} =$$

$$\begin{bmatrix}\phi x \\ -y(\frac{(a+b+2-\lambda)(b-\lambda+1)}{a+1-\lambda})\end{bmatrix} = \begin{bmatrix}\phi x \\ -\frac{b-(\lambda-1)}{a-(\lambda-1)}(\phi y)\end{bmatrix} = \phi\begin{bmatrix}x \\ -\frac{b-(\lambda-1)}{a-(\lambda-1)}y\end{bmatrix}$$

From (4.9), we can see that $[x, -\frac{b-(\lambda-1)}{a-(\lambda-1)}y]$ is an eigenvector of $L(G(a, b))$. □

Now this Theorem shows that for every eigenvalue in $L(G(a, b))$ there exists a related eigenvalue with modulated eigenvectors. However, this theorem does not guarantee the modulated eigenvectors are still a basis for ϕ. The following theorem shows that the modulation does not affect the basis.

Theorem 4.2. *If $\{v_1, ..., v_k\}$ is a basis for the eigenspace of λ, then $\{w_1, ..., w_k\}$ is a basis for the eigenspace of ϕ, where $v_j = [x_j, y_j]$ and $w_j = [x_j, my_j]$ and $m = -\frac{b-(\lambda-1)}{a-(\lambda-1)}$, such that $b \neq \lambda - 1$ or $a \neq \lambda - 1$.*

Proof. Assume that $\mathcal{H}$ is a basis of the eigenspace of λ. Thus $\mathcal{H}$ is a linearly independent list of n vectors:

$$\mathcal{H} = \{[x_1, y_1], ..., [x_n, y_n]\}. \tag{4.5}$$

Let $\mathcal{H}'$ be a set of of n modulated vectors,

$$\mathcal{H}' = \{[x_1, my_1], ..., [x_n my_n]\}. \tag{4.6}$$

to show that $\mathcal{H}'$ is linearly independent we can form the following:

$$\delta_1[x_1, my_1] + \cdots + \delta_n[x_n, my_n] = 0. \tag{4.7}$$

From this we can form the following system of equations:

$$\begin{cases} \delta_1 x_1 + \cdots + \delta_n x_n = 0 \\ \delta_1 m y_1 + \cdots + \delta_n m y_n = 0 \end{cases} = \begin{cases} \delta_1 x_1 + \cdots + \delta_n x_n = 0 \\ m(\delta_1 y_1 + \cdots + \delta_n y_n) = 0 \end{cases} \tag{4.8}$$

by the zero product property of multiplication, either m or $\delta_1 y_1 + \cdots + \delta_n y_n$ equals 0. However, m must be nonzero from the theorem, so $\delta_1 y_1 + \cdots + \delta_n y_n = -$. Now since $\mathcal{H}$ is linearly independent if,

$$\gamma_1[x_1, y_1] + \cdots + \gamma_n[x_n, y_n] = 0 \iff \begin{cases} \gamma_1 x_1 + \cdots + \gamma_n x_n = 0 \\ \gamma_1 y_1 + \cdots + \gamma_n y_n = 0 \end{cases} \tag{4.9}$$

then,

$$\gamma_1 = \cdots = \gamma_n = 0. \tag{4.10}$$

Therefore, we can form,

$$\delta_1 y_1 + \cdots + \delta_n y_n = 0 = \gamma_1 y_1 + \cdots + \gamma_n y_n \implies \delta_1 = \gamma_1 = \cdots = \delta_n = \gamma_n = 0. \tag{4.11}$$

Thus, $\mathcal{H}'$ is linearly independent. Now we need to show the the modulation on $\mathcal{H}'$ leads back to $\mathcal{H}$. So, now assume that $\mathcal{H}'$ is a basis of the eigenspace of ϕ. Thus, $\mathcal{H}'$ is a list of n linearly independent vectors,

$$\mathcal{H}' = \{[x_1, my_1], ..., [x_n my_n]\}. \tag{4.12}$$

Let $\mathcal{H}$ be a set of n modulated vectors,

$$\mathcal{H} = \{[x_n, m'y_n], ..., [x_n, m'y_n]\} \tag{4.13}$$

where $m' = \frac{1}{m}$. To show that $\mathcal{H}$ is linearly independent we can form the following,

$$\gamma_1[x_1, m'y_1] + \cdots + \gamma_n[x_n, m'y_n] = 0 \iff \begin{cases} \gamma_1 x_1 + \cdots + \gamma_n x_n = 0 \\ \gamma_1 m' y_1 + \cdots + \gamma_n m' y_n = 0 \end{cases} \tag{4.14}$$

therefore,

$$\begin{cases} \gamma_1 x_1 + \cdots + \gamma_n x_n = 0 \\ m'(\gamma_1 y_1 + \cdots + \gamma_n y_n) = 0 \end{cases} \tag{4.15}$$

by the zero product property of multiplication, either m' or $\gamma_1 y_1 + \cdots + \gamma_n y_n$ equals 0. However, by the theorem m' must be nonzero, therefore $\gamma_1 y_1 + \cdots + \gamma_n y_n = 0$. Since $\mathcal{H}'$ is linearly independent, similarly to in the first case we can form,

$$\gamma_1 y_1 + \cdots + \gamma_n y_n = 0 = \delta_1 y_1 + \cdots + \delta_n y_n \implies \gamma_1 = \delta_1 = \cdots = \gamma_n = \delta_n = 0 \tag{4.16}$$

therefore, we can conclude that $\mathcal{H}$ is linearly independent. Thus the modulation does not affect the eigenspaces of λ or ϕ. □

Corollary 4.3. *The multiplicity of eigenvalues are symmetric across the $b+1$ eigenvalue.*

Proof. From Theorem 4.2, it was seen that the modulation of the eigenvectors does not affect the size of the eigenspace, thus the multiplicity of the eigenvalue stays the same after the modulation. Therefore, the multiplicity of the eigenvalues are symmetric across the $b+1$ eigenvector which is the middle eigenvector of the spectrum. □

These theorems and corollaries show that half of the eigenvalues and eigenvectors of $L(G(a,b))$ are connected to one another, thus reducing the total amount of non-negative eigenvalues that need to be shown in general to half the original amount.

5. General Formula for the Multiplicity of the Eigenvalues

In this section we explore the conjectured formula of the multiplicity for any eigenvalue of $L(G(a,b))$ and its subsequent consequences. We start with a conjecture that describes all the eigenvalues of $L(G(a,b))$,

Conjecture 5.1. *All the eigenvalues of $L(G(a,b))$ are $\{0,1,2,...,b,b+1,a+2,...,a+b,a+b+1,a+b+2\}$.*

Definition 5.1. $\binom{n}{-1} = 0 = \binom{n}{a+b+2}$

We make this definition because otherwise the sum of the multiplicities would not make sense. The following conjecture makes use of this definition.

Conjecture 5.2. *For eigenvalues less than equal to $b+1$ the multiplicity is $\binom{n}{\lambda} - \binom{n}{\lambda-1}$ and for the eigenvalues greater than $b+1$ the multiplicity is $\binom{n}{\lambda-1} - \binom{n}{\lambda}$.*

Proof. Take the eigenvalues as described in Conjecture 5.1 and split them into two sets as described by the inequalities in this conjecture. Thus, we receive $\{0,1,2,...b-1,b,b+1\}$ and $\{a+2,a+3,...,a+b,a+b+1,a+b+2\}$. Using the appropriate formula for the multiplicity, we form M as the sum of the multiplicities.

$$\begin{aligned} M_{[0,b+1]} &= -\binom{n}{-1} + \binom{n}{0} - \binom{n}{0} + \binom{n}{1} - \binom{n}{1} + \binom{n}{2} + \cdots \\ &- \binom{n}{b-2} + \binom{n}{b-1} - \binom{n}{b-1} + \binom{n}{b} - \binom{n}{b} + \binom{n}{b+1} = \binom{n}{b+1} \end{aligned} \tag{5.1}$$

It can be seen from writing the multiplicities as $-\binom{n}{\lambda-1} + \binom{n}{\lambda}$, many of the consecutive terms cancel with one another leaving us just with $\binom{n}{b+1}$. Now for the rest of the eigenvalues, we proceed in a similar way,

$$\begin{aligned} M_{[a+2,a+b+2]} &= \binom{n}{a+1} - \binom{n}{a+2} + \binom{n}{a+2} - \binom{n}{a+3} + \cdots \\ +\binom{n}{a+b-1} &- \binom{n}{a+b} + \binom{n}{a+b} - \binom{n}{a+b+1} + \binom{n}{a+b+1} - \binom{n}{a+b+2} = \binom{n}{a+1} \end{aligned} \tag{5.2}$$

So again, the consecutive terms cancel with one another leaving us just with $\binom{n}{a+1}$. Now it is clear to see that:

$$\binom{n}{b+1} + \binom{n}{a+1} = \binom{n}{a} + \binom{n}{b} \tag{5.3}$$

Therefore, $M_{[0,b+1]} + M_{[a+2,a+b+2]}$ is the total eigenspace of $L(G(a,b))$, thus it follows that for the eigenvalues less than equal to $b+1$ the multiplicity is $\binom{n}{\lambda} - \binom{n}{\lambda-1}$ and for the eigenvalues greater than $b+1$ the multiplicity is $\binom{n}{\lambda-1} - \binom{n}{\lambda}$. □

This conjecture relies on Conjecture 5.1 being true, because otherwise there could be possibly some fringe case of the multiplicities, so we can form the following theorem,

Theorem 5.2. *Conjecture 5.2 implies Conjecture 5.1.*

Proof. In the proof of Conjecture 5.2 we saw that if the eigenvalues are as described in Conjecture 5.1, then there are no other possible eigenvalues because the multiplicities of the eigenvalues sum to the total eigenspace. Therefore, Conjecture 5.2 implies Conjecture 5.1. □

6. CC^T and C^TC Matrices

We start this section with the definitions of the CC^T and C^TC matrices respectively:

Definition 6.1 (CC^T Matrix)**.** $CC^T(K) = \sum_{j\notin K}\sum_{i\in K}(K\backslash i) \cup j + (a+1)K$, where K is a b-tuple in $\mathcal{B}$.

Definition 6.2 (C^TC Matrix)**.** $C^TC(T) = \sum_{j\notin T}\sum_{i\in T}(T\backslash i) \cup j + (b+1)T$, where T is an a-tuple in $\mathcal{A}$.

These definitions describe how CC^T and CC^T transform their respective sized tuples. Interestingly the eigenvalues and eigenvectors of the CC^T and C^TC matrices are connected to the eigenvalues and eigenvectors of $L(G(a,b))$, so we can form the following theorem:

Theorem 6.3. *Given a $\lambda \in \sigma(G)$, there is a corresponding $\mu \in \sigma(CC^T)$ and $\mu \in \sigma(C^TC)$ such that $\mu = (\lambda-(a+1))(\lambda-(b+1))$, where $\lambda \neq a+1$ or $\lambda \neq b+1$.*

Proof. Given the block matrix in $L(G(a,b))$ assume that λ is an eigenvalue with a block eigenvector $[x,y]$ then,

$$\begin{bmatrix}(a+1)I & C \\ C^T & (b+1)I\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \lambda\begin{bmatrix}x \\ y\end{bmatrix} = \begin{cases}(a+1)x + Cy = \lambda x \\ C^Tx + (b+1)y = \lambda y\end{cases} \tag{6.1}$$

Therefore, it follows that,

$$\begin{cases}Cy = (\lambda-(a+1))x \\ C^Tx = (\lambda-(b+1))y\end{cases} = \begin{cases}\frac{1}{\lambda-(a+1)}Cy = x \\ \frac{1}{\lambda-(b+1)}C^Tx = y\end{cases} \tag{6.2}$$

Then if we make some substitutions to form CC^T and C^TC, we can see that,

$$
\begin{cases} \frac{1}{(\lambda-(a+1))(\lambda-(b+1))} CC^T x = x \\ \frac{1}{(\lambda-(a+1))(\lambda-(b+1))} C^T C y = y \end{cases} = \begin{cases} CC^T x = (\lambda-(a+1))(\lambda-(b+1))x \\ C^T C y = (\lambda-(a+1))(\lambda-(b+1))y \end{cases} \tag{6.3}
$$

Therefore, given a λ in $L(G(a,b))$, we see that $(\lambda-(a+1))(\lambda-(b+1)) = \mu$ is an eigenvalue in CC^T and C^TC with the eigenvectors of x and y respectively. □

As a result of this theorem, in the search for eigenvalues and eigenvectors we can just search in CC^T as it has the same eigenvectors and a similar eigenvalues. This shift in searching grounds is particularly nice as the CC^T matrix is smaller and has a nice formula to describe the transformation to a specific tuple. In addition to this, as seen in equation (6.2), given the x portion of a block eigenvector we form the respective y portion using C^T, so this makes the search even simpler as we are now only looking for the smaller portion of the eigenvector.

7. Eigenvalue 2 and $a+2$

By Theorem 4.1, we know that 2 and $a+2$ are connected to one another with the same multiplicity, since $\phi = a+b+2-\lambda \implies a+4-2 = a+2$. We also know that the eigenvectors of these two eigenvalues are connected to one another and that they have the same x portion in the block eigenvector. So we can form this first conjecture which posits a general formation for the x portion of the block eigenvector.

Conjecture 7.1. *The b^{th}-dimensional signed-cross-polytope is an eigenvector of CC^T with an eigenvalue $\mu = a+1-b$.*

The above conjecture is near a proof, but for a finite case it is much easier to prove, so we form the following theorem.

Theorem 7.1. *The 2-dimensional signed-cross-polytope is an eigenvector of CC^T of $L(G(a,2))$ with an eigenvalue $\mu = a-1$.*

Proof. Take $x_1, x_2, x_3, x_4 \in \{1, ..., n\}$ and form,

$$
(x_1, -x_2) * (x_3, -x_4) \tag{7.1}
$$

this results in $2b = 2*2 = 4$ subsets of length $b = 2$,

$$
(x_1, x_3), -(x_1, x_4), -(x_2, x_3), (x_2, x_4) \tag{7.2}
$$

So,

$$
\begin{aligned}
(7.3)\quad & CC^T((x_1,x_3),-(x_1,x_4),-(x_2,x_3),(x_2,x_4)) = \\
& \sum_{j\notin S}\sum_{i\in S}((x_1,x_3)\backslash i)\cup j + (a+1)(x_1,x_3) - ((x_1,x_4)\backslash i)\cup j - (a+1)(x_1,x_4) \\
& - ((x_2,x_3)\backslash i)\cup j - (a+1)(x_2,x_3) + ((x_2,x_4)\backslash i)\cup j + (a+1)(x_2,x_4) = \\
& \sum_{j\notin S}(x_1)\cup j + (x_3)\cup j + (a+1)(x_1,x_3) - (x_1)\cup j - (x_4)\cup j - (a+1)(x_1,x_4) \\
& - (x_2)\cup j - (x_3)\cup j - (a+1)(x_2,x_3) + (x_2)\cup j + (x_4)\cup j + (a+1)(x_2,x_4) = \\
& \sum_{j\notin S}(x_1)\cup j + (x_1,x_3) - (x_1)\cup j - (x_1,x_4) + (x_3)\cup j + (x_1,x_3) - (x_3)\cup j - (x_2,x_3) + \\
& (x_2)\cup j + (x_2,x_4) - (x_2)\cup j - (x_2,x_3) + (x_4)\cup j + (x_2,x_4) - (x_4)\cup j - (x_1,x_4) + \\
& (a-1)((x_1,x_3) - (x_1,x_4) - (x_2,x_3) + (x_2,x_4)) = \\
& (a-1)((x_1,x_3) - (x_1,x_4) - (x_2,x_3) + (x_2,x_4))
\end{aligned}
$$

Thus $a+1-b = a-1$ is an eigenvalue of CC^T with an eigenvector of a 2-dimensional signed-cross-polytope. $\square$

Since, $a-1$ is an eigenvalue of CC^T of $L(G(a,2))$ with an eigenvector of 2-dimensional signed-cross-polytope, we can make the following theorem,

Theorem 7.2. *$b = 2$ is an eigenvalue of $L(G(a,2))$ with a block eigenvector $[x,y]$ where x is a 2-dimensional signed-cross-polytope and $y = -C^T x$.*

Proof. Take $[x,y] = [x,-C^T x]$ as a vector described in the theorem, then

$$
\begin{aligned}
(7.4)\quad & \begin{bmatrix}(a+1)I & C\\ C^T & 3I\end{bmatrix}\begin{bmatrix}x\\ -C^T x\end{bmatrix} = \begin{bmatrix}(a+1)x - CC^T x\\ C^T x - 3C^T x\end{bmatrix} \\
(7.5)\quad & = \begin{bmatrix}(a+1)x - (a-1)x\\ -2C^T x\end{bmatrix} \\
(7.6)\quad & = \begin{bmatrix}x(a+1-a+1)\\ -2C^T x\end{bmatrix} \\
(7.7)\quad & = \begin{bmatrix}2x\\ -2C^T x\end{bmatrix} = 2\begin{bmatrix}x\\ -C^T x\end{bmatrix}
\end{aligned}
$$

Therefore, $[x,-C^T x]$, where x is a 2-dimensional signed-cross-polytope, is an eigenvector of $L(G(a,2))$ with an eigenvalue of $2 = b$. $\square$

This result isn't surprising as Theorem 6.3 establishes such a strong connection between the CC^T and $C^T C$ matrices with $L(G(a,b))$ that a slight extension of the results of CC^T lead right into a seminal result for $L(G(a,2))$. In fact, as presented in Conjecture 7.1, the b^{th}-dimensional signed-cross-polytope should lead nicely into showing that $\lambda = b \in \sigma(L(G(a,b)))$; if it can be shown that $\mu = a+1-b$ is always an eigenvalue of CC^T with the signed-cross-polytope as an eigenvector, showing that $\lambda = b$ is always an eigenvalue of $L(G(a,b))$ will be nearly a trivial extension. The result presented in Theorem 7.1 can be extended even further with Theorem 4.1 to show the existence of another eigenvalue.

Theorem 7.3. *$a+2$ is an eigenvalue of $L(G(a,2))$ with a block eigenvector where x is a 2-dimensional signed-cross-polytope and $y=(a-1)C^Tx$.*

Proof. By Theorem 4.1, since $[x,-C^Tx]$ is an eigenvector of $L(G(a,2))$ with an eigenvalue $2=b$, then $[x,-\frac{2+1-2}{a+1-2}(-C^Tx)]=[x,\frac{1}{a-1}C^Tx]$ is an eigenvector of $L(G(a,2))$ with an eigenvalue $\phi=a+2+2-2=a+2$. This can easily be seen with the block matrix as well,

$$\begin{bmatrix}(a+1)I & C\\ C^T & 3I\end{bmatrix}\begin{bmatrix}x\\ \frac{1}{a-1}C^Tx\end{bmatrix}=\begin{bmatrix}(a+1)x+\frac{1}{a-1}CC^Tx\\ C^Tx-3(\frac{1}{a-1}C^Tx)\end{bmatrix} \tag{7.8}$$

$$=\begin{bmatrix}(a+1)x+\frac{(a-1)}{a-1}x\\ C^Tx(1+\frac{3}{a-1})\end{bmatrix} \tag{7.9}$$

$$=\begin{bmatrix}x(a+1+1)\\ C^Tx(\frac{a-1+3}{a-1})\end{bmatrix} \tag{7.10}$$

$$=\begin{bmatrix}(a+2)x\\ (a+2)\frac{1}{a-1}C^Tx\end{bmatrix}=(a+2)\begin{bmatrix}x\\ \frac{1}{a-1}C^Tx\end{bmatrix} \tag{7.11}$$

Therefore, $[x,\frac{1}{a-1}C^Tx]$, where x is a 2-dimensional signed-cross-polytope, is an eigenvector of $L(G(a,2))$ with an eigenvalue of $\phi=a+2$. □

8. Further Conjectures

Through various forms of analysis and observations about the eigenvectors and eigenvalues we can write the following conjectures about $L(G(a,b))$:

Conjecture 8.1. *There are $2b+3$ distinct eigenvalues for $L(G(a,b))$.*

Proof. Given the eigenvalues as they are described in Conjecture 5.1, $\{0,1,2,...,b,b+1,a+2,...,a+b,a+b+1,a+b+2\}$. From $[0,b]$ there are $b+1$ elements and by Theorem 4.1 $[0,b]$ correspond to $[a+2,a+b+2]$, so there are an additional $b+1$ element. With the addition of the $b+1$ eigenvalue that brings the total amount of distinct eigenvalues to $b+1+b+1+1=2b+3$, just as desired. □

Conjecture 8.2. *The eigenvalues of $L(G(a,2))$ are $\{0,1,2,3,a+2,a+3,a+4\}$ with multiplicities $\{1,\binom{n}{1}-1,\binom{n}{2}-\binom{n}{1},\binom{n}{3}-\binom{n}{2},\binom{n}{a+1}-\binom{n}{a+2},\binom{n}{a+2}-\binom{n}{a+3},1\}$, respectively.*

Conjecture 8.3. *The eigenvectors for eigenvalue $\mu=2(a+2-b)=2a$ in CC^T of $L(G(a,2))$ are $K_{b,a+1}$ complete partite graphs. This eigenvalue corresponds to $\lambda=b-1$ in $L(G(a,2))$.*

Conjecture 8.4. *Given $a,b,c,d\in\mathbb{N}$, if $a+b=c+d$ and $c>a>b>d$, then $\sigma(G(c,d))\subset\sigma(G(a,b))$*

Proof. According to Conjecture 5.1, which states that the eigenvalues of $L(G(a,b))$ are $\{0,1,2,..,b,b+1,a+2,...,a+b,a+b+1,a+b+2\}$, we know that $\sigma(G(c,d))=\{0,1,2,\cdots,d,d+1,c+2,\cdots,c+d,c+d+1,c+d+2\}$ and $\sigma(G(a,b))=\{0,1,2,\cdots,b,b+1,a+2,\cdots,a+b,a+b+1,a+b+2\}$, Since, $b>d$, we know that, $\{0,1,2,\cdots,d,d+1\}\subset\{0,1,2,\cdots,b\}$.

Now it needs to be shown that $\{c+2,\cdots,c+d,c+d+1,c+d+2\}\subset\{b+1,a+2,\cdots,a+b,a+b+1,a+b+2\}$ So, take $a+b=c+d=z$, then $\{c+2,\cdots,z,z+1,z+$

$2\} \subset \{b+1, a+2, \cdots, z, z+1, z+2\}$; however, we can write $c+2 = c+d+2-(d)$ and $a+2 = a+b+2-(b)$, so $\{z+2-(d), \cdots, z+2-(d-(d-1)), z+2-(d-d)\} \subset \{b+1, z+2-(b), \cdots, z+2-(b-(b-1)), z+2-(b-b)\}$. Since $d < b$, all the $(z+2)-d_i's$ have an equivalent $(z+2)-b_i$, thus, $\{c+2, \cdots, c+d, c+d+1, c+d+2\} \subset \{b+1, a+2, \cdots, a+b, a+b+1, a+b+2\}$.

Therefore, $\sigma(G(c,d)) \subset \sigma(G(a,b))$. □

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