Math 4329: Numerical Analysis Chapter 03:Secant Method

Natasha S. Sharma, PhD

What we covered so far with numerical root finding methods

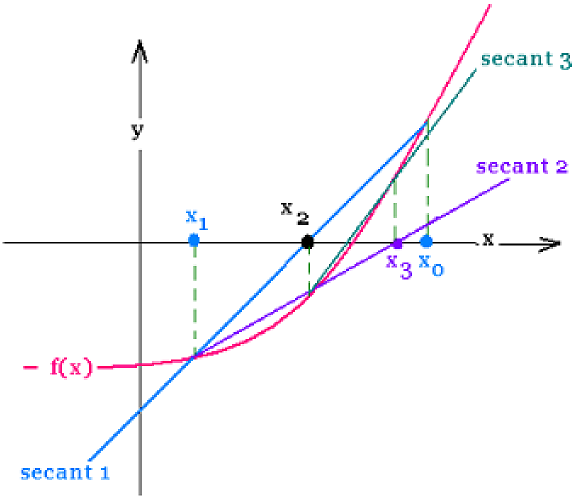
- Bisection Method is slow but helps to figure out the location of the root.
  The speed of convergence is linear since

$$|\alpha - c_{n+1}| \leq \frac{1}{2}|\alpha - c_n|$$

- Newton's Method is also slow to begin with but speedily converges to the root at a quadratic rate. The initial speed depends on the amplification factor M and the choice of x_0.

Basic Idea Behind Secant Method

Given x_0 and x_1, x_2 is x-intercept of the **secant** line.



Secant Method

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Secant line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$:

$$\begin{aligned} y(x) &= f(x_1) + f'(x_1)(x - x_1) \\ &= f(x_1) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_1). \end{aligned}$$

x_2 as the x-intercept of the secant line that is

$$f(x_1) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x_2 - x_1) = 0,$$

this simplifies to

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) f(x_1).$$

Generalizing, we can generate a sequence $\{x_n\}_{n \geq 1}$ where

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n), \quad n = 1, 2, \cdots$$

Example

Find the largest root of

$$f(x) = x^6 - x - 1 = 0$$

accurate within $\varepsilon = 1e-3$ using secant method.

mysecant.m

```
function x = mysecant(x0,x1,err_bd, max_it)
y0 = f(x0); y1 = f(x1);
error=1;
i =0;
while abs(error)> err_bd & i <=max_it
i = i+1;
x = x1 - (x1 -x0)*y1/(y1 -y0) ;
y=f(x) ;
x0=x1;y0=y1;
x1=x;y1=y;
error=x1-x0;
end
function value =f(x)
value = x.^6 -x-1.
```

Performance of the Secant Method

n	x_n	$f(x_n)$	$x_n - x_{n-1}$	$\alpha - x_{n-1}$
0	2	61.0	–	–
1	1.0	-1.0	-1.0	–
2	1.01612903	-9.15e-1	1.61e-2	-1.35e-1
3	1.19057777	6.57e-1	1.7e-1	-1.19e-1
4	1.11765583	-1.68e-1	-7.29e-2	-5.59e-2
5	1.13253155	-2.24e-2	1.49e-2	1.71e-2
6	1.13481681	9.54e-4	2.29e-3	2.19e-3
7	1.13472365	-5.07e-6	-9.32e-5	-9.27e-5
8	1.13472414	-1.13e-9	4.92e-7	4.92e-7
⋮	⋮	⋮	⋮	⋮
α	1.134724138			

Remarks

1. May converge slowly at first. However, as the iterates come closer to the root, the speed of convergence

Error Analysis

Assume that $f(x)$ has atleast continuous derivatives of order 2 for all x in some interval containing α and $f''(\alpha) \neq 0$.

$$\alpha - x_{n+1} = (\alpha - x_n)(\alpha - x_{n-1})\left[\frac{-f''(\eta_n)}{2f'(\psi_n)}\right].$$

Error in x_{n+1} is nearly proportional to the square of the error in x_n.
The term $\frac{-f''(\eta_n)}{2f'(\psi_n)}$ is the amplification factor. However, it depends on n. We need to make this factor independent of n. This can be achieved in the following manner:

$$\frac{-f''(c_n)}{2f'(x_n)} \approx \frac{-f''(\alpha)}{2f'(\alpha)} = M.$$

$$M = \max_{x \in [a,b]} \frac{-f''(x)}{2f'(x)}.$$

Order of Convergence

A sequence $\{x_n\}_{n\geq 0}$ converges to α with order $p \geq 1$ if $|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p, \ n \geq 0$ for some $c \geq 0$.

- $p = 1$ and $c < 1$ linear convergence (Bisection Method),
- $1 < p < 2$ superlinear convergence,
- $p = 2$ quadratic convergence (Newton's Method).

By writing $f(x)$ using the Taylor's expansion in the remainder form, we derive the superlinear convergence of the secant method.

$$|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p, \quad \text{with } p \approx 1.62.$$

Newton's Method vs. Secant Method

1. Two function evaluations of $f(x)$ and $f'(x)$ per iteration while secant needs only one evaluation. Thus, secant is faster.
2. This depends on the behavior of $f(x)$ for example, how hard is it to compute the derivative of $f(x)$.
3. Conclusion: The choice of root finding method depends on the underlying function $f(x)$ and the human factors such as efficient versus accuracy or convenience of use.